題名

重力と電磁気力

(‥相対論不変の下で‥)

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行列ベクトルについて

(1)**ベクトルとしての簡明さ**

(2)**行列としての機能性**　　　**をあわせ持つ。**

- **四元ベクトル**　　　　　<u>**行列ベクトル**</u>

$$\begin{pmatrix} ct \\ \mathbf{r} \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} ct & \\ & \mathbf{r} \end{pmatrix} = \begin{pmatrix} ct + x & y + iz \\ y - iz & ct - x \end{pmatrix}$$

行列の積⇒四元ベクトル積

$$
\begin{pmatrix} ct & \\ & \mathbf{r} \end{pmatrix}\begin{pmatrix} ct' & \\ & \mathbf{r}' \end{pmatrix}=\begin{pmatrix} ct+x & y+iz \\ y-iz & ct-x \end{pmatrix}\begin{pmatrix} ct'+x' & y'+iz' \\ y'-iz' & ct'-x' \end{pmatrix}
$$

$$
=\begin{pmatrix} (ctct'+\underline{xx'+yy'+zz'}) & [(cty'+yct')-i(\underline{\underline{zx'-xz'}})] \\ +[(ctx'+xct')-i(\underline{\underline{yz-zy'}})] & +i[(ctz'+zct')-i(\underline{\underline{xy'-yx'}}) \\ & \\ [(cty'+yct')-i(\underline{\underline{zx'-xz'}})] & (ctct'+\underline{xx'+yy'+zz'}) \\ -i[(ctz'+zct')-i(\underline{\underline{xy'-yx'}})] & -[(ctx'+xct')-i(\underline{\underline{yz'-zy'}}) \end{pmatrix}
$$

$$
=\begin{pmatrix} ctct'+\underline{\mathbf{r}\cdot\mathbf{r}'} & \\ & ct\mathbf{r}'+\mathbf{r}ct'-i(\underline{\underline{\mathbf{r}\times\mathbf{r}'}}) \end{pmatrix}
$$

ローレンツ変換⇒ローレンツ形式

*ローレンツ変換(1)

$$\begin{cases} ct' = \gamma(ct - \beta x) \\ x' = \gamma(x - \beta ct) \\ y' = y \\ z' = z \end{cases} \quad \Rightarrow$$

*ローレンツ変換(2)

$$\begin{cases} ct' + x' = \gamma(1-\beta)(ct + x) \\ ct' - x' = \gamma(1+\beta)(ct - x) \\ y' + iz' = y + iz \\ y' - iz' = y - iz \end{cases}$$

*ローレンツ形式(1)•••行列

$$\begin{pmatrix} ct' + x' & y' + iz' \\ y' - iz' & ct' - x' \end{pmatrix} = \begin{pmatrix} \gamma(1-\beta)(ct+x) & y + iz \\ y - iz & \gamma(1+\beta)(ct - x) \end{pmatrix}, \gamma_+ = \sqrt{\frac{\gamma+1}{2}}, \quad \gamma_- = \sqrt{\frac{\gamma-1}{2}}$$

$$= \begin{pmatrix} \gamma_+ - \gamma_- & 0 \\ 0 & \gamma_+ + \gamma_- \end{pmatrix} \begin{pmatrix} ct + x & y + iz \\ y - iz & ct - x \end{pmatrix} \begin{pmatrix} \gamma_+ - \gamma_- & 0 \\ 0 & \gamma_+ + \gamma_- \end{pmatrix}$$

*ローレンツ形式(2)•••行列ベクトル，$\boldsymbol{\gamma}_0 = (\gamma_- \quad 0 \quad 0)$

$$\begin{pmatrix} ct' & \\ & \mathbf{r}' \end{pmatrix} = \begin{pmatrix} \gamma_+ & \\ & -\boldsymbol{\gamma}_0 \end{pmatrix} \begin{pmatrix} ct & \\ & \mathbf{r} \end{pmatrix} \begin{pmatrix} \gamma_+ & \\ & -\boldsymbol{\gamma}_0 \end{pmatrix} = \begin{pmatrix} ct & \\ & \mathbf{r} \end{pmatrix}$$

−　−

行列ベクトルの応用 (1)

＊マクスウェル方程式($\underline{\underline{E_t}}$: 時間成分)

$$ {}^{+}\begin{pmatrix} \dfrac{\partial}{\partial ct} & \\ & \dfrac{\partial}{\partial \mathbf{r}} \end{pmatrix}^{+} {}^{-}\begin{pmatrix} \underline{\underline{E_t}} & \\ & \mathbf{E}-ic\mathbf{B} \end{pmatrix}^{+} = {}^{+}\begin{pmatrix} \dfrac{\rho}{\varepsilon_0} & \\ & -\dfrac{\mathbf{j}}{\varepsilon_0 c} \end{pmatrix}^{+} $$

$$ \Leftrightarrow \begin{cases} \mathbf{rotE}+\dfrac{\partial c\mathbf{B}}{\partial ct}=\mathbf{0} & \text{(ファラデーの式)•••(1)ベクトル} \\ div\,c\mathbf{B}=0 & \text{(磁束保存の式)•••(2)スカラー} \\ div\mathbf{E}+\dfrac{\partial \underline{\underline{E_t}}}{\partial ct}=\dfrac{\rho}{\varepsilon_0} & \text{(ガウスの式)•••(3)スカラー} \\ \mathbf{rot}c\mathbf{B}-\dfrac{\partial \mathbf{E}}{\partial ct}-\mathbf{grad}\underline{\underline{E_t}}=\dfrac{\mathbf{j}}{\varepsilon_0 c} & \text{(アンペールの式)•••(4)ベクトル} \end{cases} $$

行列ベクトルの応用 (2)

＊電磁場ーポテンシャル($\underline{\underline{E_t}}$: 時間成分)

$$
{}^{-}\begin{pmatrix} \dfrac{\partial}{\partial ct} & \\ & -\dfrac{\partial}{\partial \mathbf{r}} \end{pmatrix}^{-}
\;{}^{+}\begin{pmatrix} \phi & \\ & -c\mathbf{A} \end{pmatrix}^{+}
= {}^{-}\begin{pmatrix} \underline{\underline{E_t}} & \\ & \mathbf{E}-ic\mathbf{B} \end{pmatrix}^{+}
$$

$$
\Leftrightarrow
\begin{cases}
\underline{\underline{E_t}} = \dfrac{\partial \phi}{\partial ct} + div c\mathbf{A} \cdots (1) \\[2ex]
\qquad \left\{ E_t = \dfrac{\partial \phi}{\partial ct} + div c\mathbf{A} = 0 : \text{Lorenz gauze} \right\} \left\{ E_t = \dfrac{\partial \phi}{\partial ct} (\Leftrightarrow div c\mathbf{A} = 0) : \text{Coulomb gauze} \right\} \\[2ex]
\mathbf{E} = -\dfrac{\partial c\mathbf{A}}{\partial ct} - \mathbf{grad}\,\phi \cdots (2) \\[2ex]
c\mathbf{B} = \mathbf{rot}\,c\mathbf{A} \cdots (3)
\end{cases}
$$

行列ベクトルの応用 (3)

＊クーロン・ローレンツ力($\underline{\underline{E_t}}$: 時間成分)

$$
{}^{-}\begin{pmatrix} F_t & \\ & \mathbf{F} \end{pmatrix}^{-} = {}^{-}\begin{pmatrix} \underline{\underline{E_t}} & \\ & \mathbf{E}-ic\mathbf{B} \end{pmatrix}^{+} \; {}^{-}\begin{pmatrix} q & \\ & \dfrac{\mathbf{j}}{c} \end{pmatrix}^{-}
$$

$$
\Leftrightarrow \begin{cases} F_t = q\underline{\underline{E_t}} + \dfrac{\mathbf{j}}{c}\cdot\mathbf{E} - i\underline{\dfrac{\mathbf{j}}{c}\cdot c\mathbf{B}} & \text{(エネルギーの変化)} \\ \mathbf{F} = q\mathbf{E} + \dfrac{\mathbf{j}}{c}\underline{\underline{E_t}} + \dfrac{\mathbf{j}}{c}\times c\mathbf{B} - i\underline{\left(qc\mathbf{B} - \dfrac{\mathbf{j}}{c}\times\mathbf{E}\right)} & \text{(運動量の変化)} \end{cases}
$$

<u>下線部</u>：複素力

重力との類似 (1)

*静止状態の電荷eのときポテンシャルは

$$\phi = \frac{1}{4\pi\varepsilon_0}\frac{e}{r} \quad , \mathbf{A} = \mathbf{0}$$

*このときの電磁場は

$$\underline{\underline{E_t}} = \frac{\partial \phi}{\partial ct} + div c\mathbf{A} = 0$$

$$\mathbf{E} = -\frac{\partial c\mathbf{A}}{\partial ct} - \mathbf{grad}\,\phi = -\frac{1}{4\pi\varepsilon_0}\frac{\partial}{\partial \mathbf{r}}\left(\frac{e}{r}\right)$$

$$c\mathbf{B} = \mathbf{rot}c\mathbf{A} = \mathbf{0}$$

重力との類似 (2)

*運動している電荷$(q, \frac{\mathbf{j}}{c}) = q_0(\frac{u_t}{c}, \frac{\mathbf{u}}{c})$が受ける力は

$$\mathbf{F} = \underline{\underline{-\frac{q_0}{4\pi\varepsilon_0 c}\frac{\partial}{\partial \mathbf{r}}(\frac{e}{r})u_t}} + i\frac{q_0}{4\pi\varepsilon_0 c}\frac{\partial}{\partial \mathbf{r}}(\frac{e}{r}) \times \mathbf{u}$$

*ポテンシャル$U = \frac{G}{c^2}\frac{M}{r}$から質点$m$が受ける力は

$$\mathbf{f} = -m_0\frac{\partial U}{\partial \mathbf{r}} \quad - m_0 \underset{=}{\gamma}\frac{\partial U}{\partial \mathbf{r}} = \underline{\underline{-\frac{Gm_0}{c^2}\frac{\partial}{\partial \mathbf{r}}(\frac{M}{r})u_t}}$$

$$(\underset{=}{\gamma} = \frac{u_t}{c} \quad 1)$$

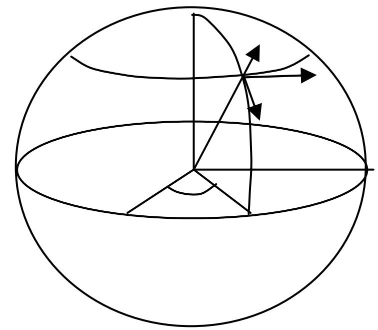
重力への応用(1)

*重力とポテンシャル

$$\begin{cases} f_t = -\dfrac{Gm_0}{c^3}\dfrac{\partial}{\partial \mathbf{r}}(\dfrac{M}{r})\cdot \mathbf{u} & \text{(エネルギーの変化)} \\ \mathbf{f} = -\dfrac{Gm_0}{c^3}\dfrac{\partial}{\partial \mathbf{r}}(\dfrac{M}{r})u_t + \underline{i\dfrac{Gm_0}{c^3}\dfrac{\partial}{\partial \mathbf{r}}(\dfrac{M}{r})\times \mathbf{u}} & \text{(運動量の変化)} \end{cases}$$

重力への応用 (2)

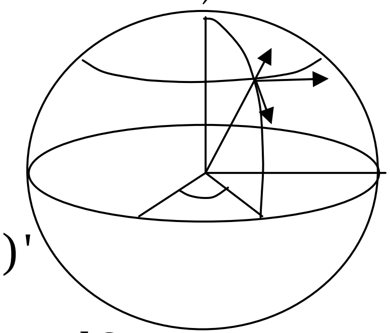
*重力(惑星)の方程式



$$
\begin{cases}
\dfrac{d^2ct}{d\tau^2} = -\dfrac{M_G}{r^2}\dfrac{dr}{d\tau}\dfrac{dct}{d\tau} \cdots (1)_{ct} \\
\dfrac{d^2r}{d\tau^2} = -\dfrac{M_G}{r^2}(\dfrac{dct}{d\tau})^2 + \dfrac{1}{r}(r\dfrac{d\theta}{d\tau})^2 + \dfrac{1}{r}(r\sin\theta\dfrac{d\phi}{d\tau})^2 \cdots (2)_r \\
\dfrac{d}{d\tau}(r\dfrac{d\theta}{d\tau}) = -i\dfrac{M_G}{r^2}(r\sin\theta\dfrac{d\phi}{d\tau})\dfrac{dct}{d\tau} - \dfrac{1}{r}\dfrac{dr}{d\tau}(r\dfrac{d\theta}{d\tau}) + \cos\theta\dfrac{d\phi}{d\tau}(r\sin\theta\dfrac{d\phi}{d\tau}) \cdots i(3)_\theta \\
\dfrac{d}{d\tau}(r\sin\theta\dfrac{d\phi}{d\tau}) = i\dfrac{M_G}{r^2}(r\dfrac{d\theta}{d\tau})\dfrac{dct}{d\tau} - \dfrac{1}{r}\dfrac{dr}{d\tau}(r\sin\theta\dfrac{d\phi}{d\tau}) - \cos\theta(r\dfrac{d\theta}{d\tau})\dfrac{d\phi}{d\tau} \cdots i(4)_\phi
\end{cases}
$$

重力への応用 (3)

*重力(惑星)の方程式($\theta = \dfrac{\pi}{2} - i\Theta$: 赤道上, Θ : 虚角で実方程式)

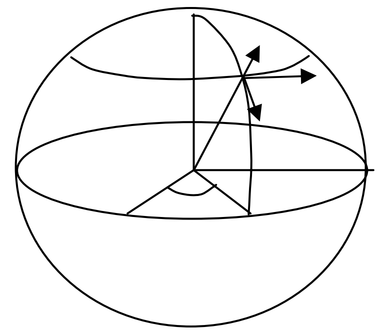


$$
\begin{cases}
\dfrac{d^2ct}{d\tau^2} = -\dfrac{M_G}{r^2}\dfrac{dr}{d\tau}\dfrac{dct}{d\tau} \cdots (1)' \\
\dfrac{d^2r}{d\tau^2} = -\dfrac{M_G}{r^2}\left(\dfrac{dct}{d\tau}\right)^2 + \dfrac{1}{r}\left\{\left(r\cosh\Theta\dfrac{d\phi}{d\tau}\right)^2 - \left(r\dfrac{d\Theta}{d\tau}\right)^2\right\} \cdots (2)' \\
\dfrac{d}{d\tau}\left(r\dfrac{d\Theta}{d\tau}\right) = \left(\dfrac{M_G}{r^2}\dfrac{dct}{d\tau} - \sinh\Theta\dfrac{d\phi}{d\tau}\right)\left(r\cosh\Theta\dfrac{d\phi}{d\tau}\right) - \dfrac{1}{r}\dfrac{dr}{d\tau}\left(r\dfrac{d\Theta}{d\tau}\right) \cdots (3)' \\
\dfrac{d}{d\tau}\left(r\cosh\Theta\dfrac{d\phi}{d\tau}\right) = \left(\dfrac{M_G}{r^2}\dfrac{dct}{d\tau} - \sinh\Theta\dfrac{d\phi}{d\tau}\right)\left(r\dfrac{d\Theta}{d\tau}\right) - \dfrac{1}{r}\dfrac{dr}{d\tau}\left(r\cosh\Theta\dfrac{d\phi}{d\tau}\right) \cdots (4)'
\end{cases}
$$

重力への応用（4）

*惑星（円軌道）の方程式$(r, \theta = \frac{\pi}{2} - i\Theta$：一定$)$

$$\begin{cases} \dfrac{d^2 ct}{d\tau^2} = 0 \cdots (1)' \\ \dfrac{M_G}{r^2}\left(\dfrac{dct}{d\tau}\right)^2 - \dfrac{1}{r}\left(r\cosh\Theta\dfrac{d\phi}{d\tau}\right)^2 = 0 \cdots (2)' \\ \dfrac{M_G}{r^2}\dfrac{dct}{d\tau} - \sinh\Theta\dfrac{d\phi}{d\tau} = 0 \cdots (3)' \\ \dfrac{d}{d\tau}\left(r\cosh\Theta\dfrac{d\phi}{d\tau}\right) = 0 \cdots (4)' \end{cases}$$



重力への応用 (5)

*惑星(円軌道)の方程式$(r, \theta = \frac{\pi}{2} - i\Theta$: 一定$)$

$$
\begin{cases}
\dfrac{dct}{d\tau} = C \cdots (1)' \\
\dfrac{M_G}{r^2} - \dfrac{1}{r}\left(r\cosh\Theta \dfrac{d\phi}{dct}\right)^2 = 0 \cdots (2)' \\
\dfrac{M_G}{r^2} - \sinh\Theta \dfrac{d\phi}{dct} = 0 \cdots (3)' \\
r\cosh\Theta \dfrac{d\phi}{d\tau} = D \cdots (4)'
\end{cases}
$$