~~四元ベクトルの新記法と相対論形式~~

New Notation and Relativistic Form

of

the 4-dimensional Vector in Time-Space

Ｙｏｓｈｉｏ　TAKEMOTO

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In this paper, we discuss the new notation and the relativistic form of the 4-dimensional vector and their usefulness.

Contents:

In §1 we define the **matrix-vector** and the **relativistic form** and we show their examples.

We use the u(1)-matrix form as an expression of the 4-dimensional vectors and adopt two notation.

The one is the matrix-vector which has simplicity of vector and function of matrix and we get the calculation which contained all products (to multiple scalar or vector by number, scalar product and vector product) in a 3-dimensional space.

The other one is a relativistic form[5] which is the simple expression of relativity and we can get a useful and good guide by it.

In §2 we exemplify the **modification** of Maxwell's equations simply.

We introduce the time component of electric field and modify Maxwell's equations from the point of view that the vector and scalar potential satisfy the wave equations,[1] and we rewrite these equations of Maxwell's with the use of the matrix-vector and relativistic form for the following discussion

In §3 we exemplify the 4-dimensional Lorentz force with **complex component**.

The usual Lorentz force is the 3-dimensional force which acts on the moving charge. We rewrite this Lorentz force with the use of the matrix-vector and relativistic form, and then we can get the 4-dimensional complex force which acts on the moving charge.[5)]

§1. Introduction

In the previous paper, we considered the 3-dimensional electric and magnetic field as a complex 3-dimensional field $\mathbb{E} = \mathbf{E} - ic\mathbf{B}$ and we add the time component E_t in this field.

As a result, it becomes a 4-dimensional complex vector field and we can modify Maxwell's equations with the use of the u(1)-matrix like this;

$$\begin{pmatrix} \frac{\rho}{\varepsilon_0} - \frac{j_x}{\varepsilon_0 c} & -\frac{j_y}{\varepsilon_0 c} - i\frac{j_z}{\varepsilon_0 c} \\ -\frac{j_y}{\varepsilon_0 c} + i\frac{j_z}{\varepsilon_0 c} & \frac{\rho}{\varepsilon_0} + \frac{j_x}{\varepsilon_0 c} \end{pmatrix} = \begin{pmatrix} \partial ct + \partial x & \partial y + i\partial z \\ \partial y - i\partial z & \partial ct - \partial x \end{pmatrix} \begin{pmatrix} E_t + (E_x - icB_x) & (E_y - icB_y) + i(E_z - icB_z) \\ (E_y - icB_y) - i(E_z - icB_z) & E_t - (E_x - icB_x) \end{pmatrix}$$

.

And we can reform this matrix equation by the matrix-vector and the relativistic form;

$${}^{+}\begin{pmatrix} \frac{\rho}{\varepsilon_0} & \\ & -\frac{\mathbf{j}}{\varepsilon_0 c} \end{pmatrix}^{+} = {}^{+}\begin{pmatrix} \partial ct & \\ & \partial \mathbf{r} \end{pmatrix}^{+} {}^{-}\begin{pmatrix} E_t & \\ & \mathbf{E} - ic\mathbf{B} \end{pmatrix}^{+},$$

and then it gives us a good view and guide.

(1) Definition of the matrix-vector and its calculation.

We identify the 4-dimensional vector $\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \in \mathbb{R}^4$ and the $u(1)-$matrix

$\begin{pmatrix} ct + x & y + iz \\ y - iz & ct - x \end{pmatrix}$、 $u(1) = \left\{ X \in M(2, \mathbb{C}) / X^* = X \right\}$.

And we adopt a new notation $\begin{pmatrix} ct & \\ & \mathbf{r} \end{pmatrix}$ ($\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$) as this $u(1)-$matrix and we call its vector (which is represented by matrix) **the matrix-vector**, and this notation has simplicity of vector and function of matrix.

We have **a simple rule for calculation** in the matrix-vector.

$$\begin{pmatrix} ct & \\ & \mathbf{r} \end{pmatrix}\begin{pmatrix} ct' & \\ & \mathbf{r}' \end{pmatrix} = \begin{pmatrix} ctct' + \underline{\mathbf{r}\cdot\mathbf{r}'} & \\ & ct\mathbf{r}' + \mathbf{r}ct' - i(\underline{\underline{\mathbf{r}\times\mathbf{r}'}}) \end{pmatrix}$$

The underlined sign “·” means scalar product and “×” vector product.

The reason for the statement above is that the equation

$$\begin{pmatrix} ct+x & y+iz \\ y-iz & ct-x \end{pmatrix}\begin{pmatrix} ct'+x' & y'+iz' \\ y'-iz' & ct'-x' \end{pmatrix}$$

$$= \begin{pmatrix} (ctct' + \underline{xx'+yy'+zz'}) + [(ctx'+xct') - i(\underline{\underline{yz-zy'}})] & [(cty'+yct') - i(\underline{\underline{zx'-xz'}})] + i[(ctz'+zct') - i(\underline{\underline{xy'-yx'}})] \\ [(cty'+yct') - i(\underline{\underline{zx'-xz'}})] - i[(ctz'+zct') - i(\underline{\underline{xy'-yx'}})] & (ctct' + \underline{xx'+yy'+zz'}) - [(ctx'+xct') - i(\underline{\underline{yz'-zy'}})] \end{pmatrix}$$

holds.

(2) Translation by relativity.

When a particle move to the x-direction at the speed v_x, then we have the relativity relation.

We put $\gamma = \dfrac{1}{\sqrt{1-(\frac{\mathbf{v_x}}{c})^2}} = \cosh\Theta$ and $\gamma\boldsymbol{\beta}_\mathbf{x} = \dfrac{\frac{\mathbf{v_x}}{c}}{\sqrt{1-(\frac{\mathbf{v_x}}{c})^2}} = \sinh\Theta$

$$\begin{cases} ct' = \gamma(ct - \boldsymbol{\beta}_x x) \\ x' = \gamma(x - \boldsymbol{\beta}_x ct) \\ y' = y \\ z' = z \end{cases}$$

And we can rewrite this relation with matrix form.

$$\begin{pmatrix} ct'+x' & y'+iz' \\ y'-iz' & ct'-x' \end{pmatrix} = \begin{pmatrix} \gamma(1-\boldsymbol{\beta}_x)(ct+x) & y+iz \\ y-iz & \gamma(1+\boldsymbol{\beta}_x)(ct-x) \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_+ - \gamma_- & 0 \\ 0 & \gamma_+ + \gamma_- \end{pmatrix} \begin{pmatrix} ct+x & y+iz \\ y-iz & ct-x \end{pmatrix} \begin{pmatrix} \gamma_+ - \gamma_- & 0 \\ 0 & \gamma_+ + \gamma_- \end{pmatrix}$$

where $\gamma_+ = \sqrt{\dfrac{\gamma+1}{2}} = \cosh\dfrac{\Theta}{2}$ 、 $\gamma_- = \sqrt{\dfrac{\gamma-1}{2}} = \sinh\dfrac{\Theta}{2}$

Then we have a relation of **matrix-vector**.

$$\begin{pmatrix} ct' & \\ & \mathbf{r}' \end{pmatrix} = \begin{pmatrix} \gamma_+ & \\ & -\boldsymbol{\gamma}_0 \end{pmatrix} \begin{pmatrix} ct & \\ & \mathbf{r} \end{pmatrix} \begin{pmatrix} \gamma_+ & \\ & -\boldsymbol{\gamma}_0 \end{pmatrix} 、 \boldsymbol{\gamma}_0 = \gamma_- \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

More generally, when a particle moves at the speed **v** with direction cosine (A, B, C), then we have the relativity relation.[5)]

$$\begin{cases} ct' = \gamma ct - \gamma\beta Ax - \gamma\beta By - \gamma\beta Cz \\ x' = -\gamma\beta Act + \{1+(\gamma-1)A^2\}x + (\gamma-1)ABy + (\gamma-1)CAz \\ y' = -\gamma\beta Bct + (\gamma-1)ABx + \{1+(\gamma-1)B^2\}y + (\gamma-1)BCz \\ z' = -\gamma\beta Cct + (\gamma-1)ACx + (\gamma-1)BCy + \{1+(\gamma-1)C^2\}z \end{cases}$$

And we can rewrite this relation with matrix form.

$$\begin{pmatrix} ct'+x' & y'+iz' \\ y'-iz' & ct'-x' \end{pmatrix} = \begin{pmatrix} \gamma_+ - \gamma_- A & -\gamma_-(B+iC) \\ -\gamma_-(B-iC) & \gamma_+ + \gamma_- A \end{pmatrix} \begin{pmatrix} ct+x & y+iz \\ y-iz & ct-x \end{pmatrix} \begin{pmatrix} \gamma_+ - \gamma_- A & -\gamma_-(B+iC) \\ -\gamma_-(B-iC) & \gamma_+ + \gamma_- A \end{pmatrix}$$

Then we have a representation of **matrix-vector**.

$$\begin{pmatrix} ct' & \\ & \mathbf{r}' \end{pmatrix} 、 \boldsymbol{\gamma}_0 = \gamma_- \begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_+ & \\ & \underline{-\boldsymbol{\gamma}_0} \end{pmatrix} \begin{pmatrix} ct & \\ & \mathbf{r} \end{pmatrix} \begin{pmatrix} \gamma_+ & \\ & \underline{-\boldsymbol{\gamma}_0} \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_+ ct - \boldsymbol{\gamma}_0 \cdot \mathbf{r} & \\ & \gamma_+ \mathbf{r} - \boldsymbol{\gamma}_0 ct + i\boldsymbol{\gamma}_0 \times \mathbf{r} \end{pmatrix} \begin{pmatrix} \gamma_+ & \\ & -\boldsymbol{\gamma}_0 \end{pmatrix} 、 (\boldsymbol{\gamma}_0 \times \mathbf{r}) \times \boldsymbol{\gamma}_0 = (\boldsymbol{\gamma}_0 \cdot \boldsymbol{\gamma}_0)\mathbf{r} - \boldsymbol{\gamma}_0(\mathbf{r} \cdot \boldsymbol{\gamma}_0)$$

$$=\begin{pmatrix} (\gamma_+{}^2+\boldsymbol{\gamma}_0{}^2)ct-2(\boldsymbol{\gamma}_0\cdot\mathbf{r})\gamma_+ & \\ & -2\gamma_+\boldsymbol{\gamma}_0 ct+\gamma_+{}^2\mathbf{r}+(\boldsymbol{\gamma}_0\cdot\mathbf{r})\boldsymbol{\gamma}_0-(\boldsymbol{\gamma}_0\times\mathbf{r})\times\boldsymbol{\gamma}_0 \end{pmatrix}$$

$$=\begin{pmatrix} \gamma ct-\gamma\beta(Ax+By+Cz) & \\ & -\gamma\beta\begin{pmatrix}A\\B\\C\end{pmatrix}ct+\begin{pmatrix}x\\y\\z\end{pmatrix}+(\gamma-1)\begin{pmatrix}A\\B\\C\end{pmatrix}(Ax+By+Cz) \end{pmatrix}$$

We adopt this translated **matrix-vector** as $\begin{pmatrix} ct' & \\ & \mathbf{r}' \end{pmatrix}={}^{-}\begin{pmatrix} ct & \\ & \mathbf{r} \end{pmatrix}^{-}$ after the underlined vector $-\boldsymbol{\gamma}_0$ above.

And we do by the same way. Then

$$\begin{pmatrix} \partial ct' & \\ & \partial\mathbf{r}' \end{pmatrix}=\begin{pmatrix} \gamma_+ & \\ & \underline{\boldsymbol{\gamma}_0} \end{pmatrix}\begin{pmatrix} \partial ct & \\ & \partial\mathbf{r} \end{pmatrix}\begin{pmatrix} \gamma_+ & \\ & \underline{\boldsymbol{\gamma}_0} \end{pmatrix}、\ \boldsymbol{\gamma}_0=\gamma_-\begin{pmatrix}A\\B\\C\end{pmatrix}$$

We adopt this translated **matrix-vector** as $\begin{pmatrix} \partial ct' & \\ & \partial\mathbf{r}' \end{pmatrix}={}^{+}\begin{pmatrix} \partial ct & \\ & \partial\mathbf{r} \end{pmatrix}^{+}$ after the underlined vector $+\boldsymbol{\gamma}_0$ above.

And so on ….

(3) Another example for the matrix-vector.

$\mathbf{E}$ and E_t (new term) are electric field(3-dimensional vector) and time-component(scalar),

$\mathbf{B}$ is a magnetic field(3-dimensional vector).

$\mathbf{A}$ and ϕ are a vector potential(3- dimensional vector) and a scalar potential.

$\partial\mathbf{r}=\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}$ and $\partial ct=\frac{\partial}{\partial ct}$ are differential operators.

Then we have a representation of **a matrix-vector.** [1)]

$${}^{-}\begin{pmatrix} E_t & \\ & \mathbf{E}-ic\mathbf{B} \end{pmatrix}^{+}={}^{-}\begin{pmatrix} \partial ct & \\ & -\partial\mathbf{r} \end{pmatrix}^{-}\,{}^{+}\begin{pmatrix} \phi & \\ & -c\mathbf{A} \end{pmatrix}^{+}$$

$$= {}^{-}\begin{pmatrix} \frac{\partial\phi}{\partial ct}+divc\mathbf{A} & \\ & -\frac{\partial c\mathbf{A}}{\partial ct}-\mathbf{grad}\phi-i\mathbf{rot}c\mathbf{A} \end{pmatrix}^{+}$$

We compare the components of this relation, then.

$$\begin{cases} E_t=\frac{\partial\phi}{\partial ct}+divc\mathbf{A}\cdots(1) \\ \mathbf{E}=-\frac{\partial c\mathbf{A}}{\partial ct}-\mathbf{grad}\phi\cdots(2) \\ c\mathbf{B}=\mathbf{rot}c\mathbf{A}\cdots(3) \end{cases}$$

The formulas (2) and (3) are the orthodox equations of Maxwell's.

The formula (1) is a time-component of electric field, we have a Lorenz gauge

$$E_t=\frac{\partial\phi}{\partial ct}+divc\mathbf{A}=0$$

and a Coulomb gauge

$$E_t=\frac{\partial\phi}{\partial ct}(\Leftrightarrow divc\mathbf{A}=0)$$

In this situation, when the function χ and vector function $\boldsymbol{\chi}$ satisfy the wave equation, the electric and magnetic fields ($\mathbf{E}$, E_t and $\mathbf{B}$) are invariant by the transformations,

$$\phi'=\phi+\underline{\frac{\partial\chi}{\partial ct}}+\underline{\underline{divc\boldsymbol{\chi}}}$$

$$c\mathbf{A}'=c\mathbf{A}-\underline{\underline{\frac{\partial c\boldsymbol{\chi}}{\partial ct}}}-\underline{\mathbf{grad}\chi}+\underline{\underline{i\mathbf{rot}c\boldsymbol{\chi}}}$$

And then the matrix form is

$${}^{+}\begin{pmatrix} \phi' & \\ & -c\mathbf{A}' \end{pmatrix}^{+}={}^{+}\begin{pmatrix} \phi & \\ & -c\mathbf{A} \end{pmatrix}^{+}+{}^{+}\begin{pmatrix} \partial ct & \\ & \partial\mathbf{r} \end{pmatrix}^{+}{}^{-}\begin{pmatrix} \chi & \\ & c\boldsymbol{\chi} \end{pmatrix}^{+}.$$

§2. The modified Maxwell's equations as an example for a matrix-vector

We have orthodox equations of Maxwell's in a vacuum.

$$
\begin{cases}
\mathbf{rotE}+\dfrac{\partial c\mathbf{B}}{\partial ct}=\mathbf{0} & \text{(Faraday's law of induction)}\cdots(4)\\
div\, c\mathbf{B}=0 & \text{(No existence of magnetic charges)}\cdots(5)\\
div\,\mathbf{E}=\dfrac{\rho}{\varepsilon_0} & \text{(Gauss' law)}\cdots(6)\\
\mathbf{rot}\,c\mathbf{B}-\dfrac{\partial\mathbf{E}}{\partial ct}=\dfrac{\mathbf{j}}{\varepsilon_0 c} & \text{(Ampere-Maxwell's law)}\cdots(7)
\end{cases}
$$

We assume that when $\rho=0$ and $\mathbf{j}=0$, the scalar potential ϕ and the vector potential $\mathbf{A}$ satisfy the wave equation, respectively.

Then

$$
\left(\frac{\rho}{\varepsilon_0}=\right)0=-\Box\phi \text{、} \Box= div\,\mathbf{grad}-\frac{\partial^2}{\partial ct^2}
$$

$$
=\frac{\partial^2\phi}{\partial ct^2}-div(\mathbf{grad}\phi)
$$

$$
=\frac{\partial}{\partial ct}\left(div\,c\mathbf{A}+\frac{\partial\phi}{\partial ct}\right)-div\left(\mathbf{grad}\phi+\frac{\partial c\mathbf{A}}{\partial ct}\right),\quad \frac{\partial}{\partial ct}div\,c\mathbf{A}=div\frac{\partial c\mathbf{A}}{\partial ct}
$$

$$
=\underline{\frac{\partial E_t}{\partial ct}}+div\mathbf{E}\quad\cdots(6)',
$$

and

$$
\left(\frac{\mathbf{j}}{\varepsilon_0 c}=\right)0=-\Box c\mathbf{A} \text{、} \Box=\mathbf{grad}\,div-\mathbf{rot}\,\mathbf{rot}-\frac{\partial^2}{\partial ct^2}
$$

$$
=\frac{\partial^2 c\mathbf{A}}{\partial ct^2}-\mathbf{grad}(div\,c\mathbf{A})+\mathbf{rot}(\mathbf{rot}\,c\mathbf{A})
$$

$$
=\frac{\partial}{\partial ct}\left(\mathbf{grad}\phi+\frac{\partial c\mathbf{A}}{\partial ct}\right)-\mathbf{grad}\left(div\,c\mathbf{A}+\frac{\partial\phi}{\partial ct}\right)+\mathbf{rot}(\mathbf{rot}\,c\mathbf{A})
$$

$$
,\ \frac{\partial}{\partial ct}\mathbf{grad}\phi=\mathbf{grad}\frac{\partial\phi}{\partial ct}
$$

$$
=-\frac{\partial\mathbf{E}}{\partial ct}-\underline{\mathbf{grad}E_t}+\mathbf{rot}\,c\mathbf{B}\quad\cdots(7)'.
$$

Therefore we get the modified Maxwell's equations

Theorem 1. (The modified Maxwell's equations)[1)]

$$\begin{cases} \mathbf{rotE}+\dfrac{\partial c\mathbf{B}}{\partial ct}=\mathbf{0}\cdots(4) \\ div\,c\mathbf{B}=0\cdots(5) \\ div\mathbf{E}+\underline{\dfrac{\partial E_t}{\partial ct}}=\dfrac{\rho}{\varepsilon_0}\cdots(6)' \\ \mathbf{rot}c\mathbf{B}-\dfrac{\partial\mathbf{E}}{\partial ct}-\underline{\mathbf{grad}E_t}=\dfrac{\mathbf{j}}{\varepsilon_0 c}\cdots(7)' \end{cases}$$

And a representation of a **matrix-vector** is

$${}^{+}\begin{pmatrix} \dfrac{\rho}{\varepsilon_0} & \\ & -\dfrac{\mathbf{j}}{\varepsilon_0 c} \end{pmatrix}^{+} = {}^{+}\begin{pmatrix} \partial ct & \\ & \partial\mathbf{r} \end{pmatrix}^{+}\,{}^{-}\begin{pmatrix} E_t & \\ & \mathbf{E}-ic\mathbf{B} \end{pmatrix}^{+}$$

We can realize that the electromagnetic field is complex 4-dimensional space（E_t-icB_t and $\mathbf{E}-ic\mathbf{B}$, $B_t=0$） in this situation.

§3. The 4-dimensional Lorentz force as a complex one.

$\mathbf{f}$ is a 3-dimensional force,

$\mathbf{j}$ and q are current and charge.

Then we have an orthodox Lorentz force.

$$\mathbf{f}=q\mathbf{E}+\mathbf{j}\times\mathbf{B}$$

Theorem 2. (The modified Lorentz force)[5)]

The 4-dimensional force (Minkowski's force) on a charge q_0 which moves at speed $\mathbf{v}$ in the field $(E_t,\mathbf{E},\mathbf{B})$ is as follows:

The charge and the current are $q = q_0\gamma$ and $\dfrac{\mathbf{j}}{c} = q_0\gamma\boldsymbol{\beta}$ respectively and

$$\begin{cases} f_t = \underline{qE_t} + \dfrac{\mathbf{j}}{c}\cdot\mathbf{E} & \text{(the variation of energy)} \\ \mathbf{f} = q\mathbf{E} + \underline{\dfrac{\mathbf{j}}{c}E_t} + \dfrac{\mathbf{j}}{c}\times c\mathbf{B} & \text{(the variation of momentum)} \end{cases}$$

The underlined parts are new terms.

This 4-dimensional force is a real part of a following complex one $(F_t, \mathbf{F})$. [5]

That is

$${}^{-}\begin{pmatrix} F_t & \\ & \mathbf{F} \end{pmatrix}^{-} = {}^{-}\begin{pmatrix} E_t & \\ & \mathbf{E} - ic\mathbf{B} \end{pmatrix}^{+} {}^{-}\begin{pmatrix} q & \\ & \dfrac{\mathbf{j}}{c} \end{pmatrix}^{-}$$

And we compare the components of this relation, then

$$\begin{cases} F_t = qE_t + \dfrac{\mathbf{j}}{c}\cdot\mathbf{E} - \underline{i\dfrac{\mathbf{j}}{c}\cdot c\mathbf{B}} \\ \mathbf{F} = q\mathbf{E} + \dfrac{\mathbf{j}}{c}E_t + \dfrac{\mathbf{j}}{c}\times c\mathbf{B} - \underline{i(qc\mathbf{B} - \dfrac{\mathbf{j}}{c}\times\mathbf{E})} \end{cases}$$

The underlined parts are imaginary ones.

More generally, let's assume $(u_t, \mathbf{u})$ is the 4-dimensional velocity.

That is

$$u_t = \frac{dct}{d\tau} = c\gamma\ ,\ \mathbf{u} = \frac{d\mathbf{r}}{d\tau} = c\gamma\boldsymbol{\beta}$$

Then $q = \gamma q_0 = \dfrac{q_0}{c}u_t$、 $\dfrac{\mathbf{j}}{c} = \gamma\boldsymbol{\beta}q_0 = \dfrac{q_0}{c}\mathbf{u}$ and we have following relation.

$$\frac{d}{d\tau}\,{}^{-}\left[\begin{pmatrix} \dfrac{\varepsilon}{c} & \\ & \mathbf{p} \end{pmatrix} + \underline{\frac{q_0}{c}\begin{pmatrix} \phi & \\ & c\mathbf{A} \end{pmatrix}}\right]^{-} = \frac{q_0}{c}\,{}^{-}\begin{pmatrix} E_t & \\ & \mathbf{E} - ic\mathbf{B} \end{pmatrix}^{+} {}^{-}\begin{pmatrix} u_t & \\ & \mathbf{u} \end{pmatrix}^{-}$$

$$= \frac{q_0}{c}\,{}^{-}\begin{pmatrix} E_t u_t + (\mathbf{E} - ic\mathbf{B})\cdot\mathbf{u} & \\ & E_t\mathbf{u} + (\mathbf{E} - ic\mathbf{B})u_t - i(\mathbf{E} - ic\mathbf{B})\times\mathbf{u} \end{pmatrix}^{-}$$

In this equation, we have two important subjects.

The one is the underlined part, momentum-energy matrix which does not appear in the usual equation. We deduce this equation from Lagrangian (matrix) by use of the variational method.

The other one is the complex force which is litter in the usual equation. We discuss this imaginary part (force) in the case of the gravitational force and advance its understanding in another paper

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