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Gravitational Force and the 4-dimensional Complex Force*

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Abstract

In this paper, we discuss the application of the 4-dimensional complex force as a gauge field

Contents:

In §1, we review a Kepler's law and gravitational force which is deduced by it.

In §2, we study the connection in the 4-dimensional complex force as a similar to an electromagnetic force and differential equation.

And in §3, we study property of the solution of the differential equation.

§ 1. the law of universal gravitation

In this section, we mention the Kepler's laws and the gravitational force law which is deduced from them. each other

[Kepler's laws]

i) (first law)

The planets all travel around the sun in elliptical orbits having the sun at one forcus.

Implicit in this law is the fact that the paths of the planets lie in planes containing the sun.

ii) (second law)

The radius vector from the sun to any planet sweeps out equal areas in equal times.

iii) (third law)

The ratio between the cube of the major axis r of a planet's orbit and the square of its period T of revolution has the same value for all the planets; that is,

$r^3/T^2 - C$(constant).

The second law, law of areas, follows from conservation of angular momentum.

The first law permits the deduction in which we obtain the attractive solar force on a planet varies with distance from Kepler's third law.

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For simplicity we shall assume that the planetary orbits are circles with radius r centered at the sun.

Then centripetal acceleration each planet experiences is

$$\alpha = v^2/r,$$

where v is its linear orbital speed and can be expressed in the period T and radius r

$$v = 2\pi r/T.$$

Hence the acceleration may be expressed as

$$\alpha = (2\pi r/T)^2/r = 4\pi^2 C/r^2.$$

The acceleration of each planet, and therefore the force exerted on it, varies inversely with the square of its distance from the sun.

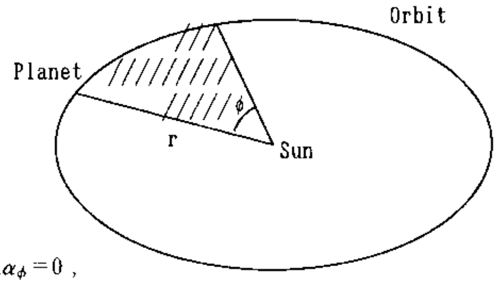
Conversely the Kepler's laws is deduced from the law of universal gravitation.

[the law of universal gravitation]
the gravitational force law between two bodies os masses M and m that are separated by the distance r are

$$F = -GMm/r^2$$

we assume that the planet move on the x-y plane and use the polar co-ordinate (r, θ); that is,

$$x = r\cos\phi ,$$
$$y = r\sin\phi .$$



Then the acceleration $a(-a_r)$ of r-direction and α_ϕ of ϕ-direction are

$$\alpha_r = -r(d\phi/dt)^2 + d^2r/dt^2$$
$$\alpha_\phi = 2(dr/dt)(d\phi/dt) + rd^2\phi/dt^2$$
$$= (1/r)d(r^2 d\phi/dt)^2/dt$$

Therefore from the law of motion i) $m\alpha = F$ and ii) $m\alpha_\phi = 0$,

i) $d^2r/dt^2 \quad r(d\phi/dt)^2 = -GM/r^2$,

ii) $d(r^2 d\phi/dt)^2/dt = 0$, i.e., $r^2 d\phi/dt = C$(constant) ,

this is the law of area.

From these i, ii), an orbit equation holds as follows :

$$d^2(1/r)/d\phi^2 + 1/r = GM/C^2 \qquad \cdots (a) .$$

Because from ii) ,

$$dr/dt = (C/r^2)dr/d\phi$$
$$= -Cd(1/r)/d\phi \qquad \cdots (b) ,$$

and

$$d^2r/dt^2 = -Cd^2(1/r)/d\phi^2(d\phi/dt)$$
$$= -(C/r)^2 d^2(1/r)/d\phi^2 \qquad \cdots (c) ,$$

hold.

The solution of (a) is an ellipse (a parabola/a hyperbola), i.e. ,

$r = (C^2/GM)/(1 + e\cos\phi)$ (Kepler s first law) ,

and constant e(>1 , -1 or <1) is an eccentricity.

From the property of an ellipse, its area S and radius of major axis r are

$$S = \pi(C^2/GM)^2/(1-e^2)^{3/2},$$

$$r_m = (C^2/GM)/(1-e^2).$$

Therefore the period T is

$$T = 2\pi(C^2/GM)^2/C(1-e^2)^{3/2}$$

$$= [2\pi/(GM)^{1/2}]\ r_m^{3/2} \quad \text{(\underline{Kepler's second law})}.$$

the correspondence between a speed at the perihelion r_1 and the orbit of the planet is as follows :

a speed (at the perihelion r_1)	an orbit
$r_1 d\phi_1/dt = (GM/r_1)^{1/2}$	a circle
$(GM/r_1)^{1/2} < r_1 d\phi_1/dt < (2GM/r_1)^{1/2}$	an ellipse
$r_1 d\phi_1/dt = (2GM/r_1)^{1/2}$	a parabola
$(2GM/r_1)^{1/2} < r_1 d\phi_1/dt$	a hyperbola

Because from the equation (a) and (c), a point r_0 such that $d^2r/dt^2 = d^2(1/r)/d\phi^2 = 0$ is

$$r_0 = C^2/GM, \quad r_0 d\phi_0/dt = (GM/r_0)^{1/2} \qquad \cdots \text{(d)}.$$

This is the same speed as a planet moving on the circle with radius r_0.

Multiply (a) by $2d(1/r)/d\phi$ and integral it, then

$$2d(1/r)/d\phi \cdot d^2(1/r)/d\phi^2 + 2/r \cdot d(1/r)/d\phi = 2GM/h^2 \cdot d(1/r)/d\phi$$

$$(d(1/r)/d\phi)^2 + 1/r^2 - 2GM/h^2 r = 1/r_1^2 - 2GM/C^2 r_1$$

where a distance $r_1 = C^2/GM(1+e) = r_0/(1+e)$ at the perihelion is the one which satisfies the equation $dr/dt = d(1/r)/d\phi = 0$.

Therefore

$$r_1 d\phi_1/dt = (1+e)^{1/2}(GM/r_1)^{1/2} \qquad \cdots \text{(e)}.$$

And a distance r_2 at the aphelion is

$$1/r_2^2 - 2GM/C^2 r_2 = 1/r_1^2 - 2GM/C^2 r_1$$

$$\therefore\ 1/r_2 + 1/r_1 = 2GM/C^2 (= 2/r_0) \qquad \cdots \text{(f)}.$$

When $r_2 \to \infty$ (ellipse) then $r_0 = 2r_1$, $e = 1$ and $r_1 d\phi_1/dt = (2GM/r_1)^{1/2}$(an escape velocity)

§ 2. The connection in the 4-dimensional force

Let's

$$F: \begin{pmatrix} F_t \\ \cdots \\ F_x \\ F_y \\ F_z \end{pmatrix} = d \begin{pmatrix} v^0 \\ -- \\ v^1 \\ v^2 \\ v^3 \end{pmatrix} /dc\tau, \qquad V: \begin{pmatrix} v^0 \\ -- \\ v^1 \\ v^2 \\ v^3 \end{pmatrix} = m_0 d \begin{pmatrix} ct \\ \\ x \\ y \\ z \end{pmatrix} /dct,$$

(where m_0 is a rest mass, τ is a proper time) ,

the Minkowski's (4 dimensional) force F and the 4-dimensional velocity V.

Theorem 1.[cf.8)]

Let' $j_0 = q_0 \gamma$, $\mathbb{J} = q_0 \gamma \mathbb{u}/c$ and $\mathbb{u}$ is the velocity of charge q_0.

A 4-dimension complex force F on a moving charge q_0 with speed $\mathbb{u}$ in the field E_0, $\mathbb{E}$, $\mathbb{B}$ is

$$\begin{cases} F_t = j_0 E_0 + \underline{\mathbb{J} \cdot \mathbb{E}} \qquad\qquad + i \ \mathbb{J} \cdot \mathbb{B} \\ \mathbb{F} = \underline{j_0} \mathbb{E} + \mathbb{J} \ E_0 + \underline{\mathbb{J} \times} \mathbb{B} + i \ (j_0 \mathbb{B} \quad \mathbb{J} \times \mathbb{E}) \end{cases}$$

where the underlined part are Lorentz force.

When the charge is stationary, i.e., $E_0 = 0$, $\mathbb{B} = \mathbb{0}$ then $F_t = \mathbb{J} \cdot \mathbb{E}$, $\mathbb{F} = j_0 \mathbb{E} - i(\mathbb{J} \times \mathbb{E})$.

Therefore the 4-dimension force F is

$$\begin{pmatrix} F_t \\ \hline F_x \\ F_y \\ F_z \end{pmatrix} = \frac{K}{r^3} \begin{pmatrix} 0 & x & y & z \\ \hline x & 0 & iz & -iy \\ y & -iz & 0 & ix \\ z & iy & ix & 0 \end{pmatrix} \begin{pmatrix} q_0 \gamma \\ \hline q_0 \gamma \beta_x \\ q_0 \gamma \beta_y \\ q_0 \gamma \beta_z \end{pmatrix} \qquad \cdots (1)$$

where K is a constant.

We assume that a gravitational force is <u>the same form as the electromagnetic force</u>, then the following theorem holds.

Theorem 2.[cf.7)]

The connection by the force received from the rest mass M_0 with distance r is

$$\frac{d}{d\tau} \begin{pmatrix} dct/d\tau \\ \hline dx/d\tau \\ dy/d\tau \\ dz/d\tau \end{pmatrix} = -\frac{M_0}{r^3} \left(\begin{pmatrix} 0 & x & y & z \\ \hline x & 0 & iz & -iy \\ y & -iz & 0 & ix \\ z & iy & -ix & 0 \end{pmatrix} \frac{dct}{d\tau} \right) \begin{pmatrix} dct/d\tau \\ \hline dx/d\tau \\ dy/d\tau \\ dz/d\tau \end{pmatrix} \qquad \gamma = \frac{dct}{d\tau}$$

and the connection in paper 7) is a real form of this one.

proof

Let Γ^i_{jk} be a connection which is torsion free and compatible with the metric g_{ij} then the covariant derivative ∇ of the vector field (v^0, v^1, v^2, v^3) is

$$\nabla u^i = dv^i + \Gamma^i_{jk} v^k \cdot dx^j, \quad (x^0, x^1, x^2, x^3) = (ct, x, y, z),$$
$$= 0.$$

Therefore

$$du^i = -\Gamma^i_{jk} u^j \cdot dx^j, \quad (\Gamma^i_{jk}) = (\Gamma^i_k)_j \in so(1, 3) \qquad \cdots (2)$$

this <u>means 4 dimentional force</u>.

we compare (1) with (2) and can assume that the gravitational force by the rest mass M_0,

$$
(\Gamma^{i}{}_{k})_{1}=-\frac{GM_0}{r^3}\left(\begin{array}{c|ccc} 0 & x & y & z \\ \hline x & 0 & iz & -iy \\ y & -iz & 0 & ix \\ z & iy & -ix & 0 \end{array}\right), \quad (\Gamma^{i}{}_{k})_{2}=(\Gamma^{i}{}_{k})_{3}=(\Gamma^{i}{}_{k})_{4}=0\ ,
$$

q. e. d.

We rewrite the above connection to the polar coordinate form by the coordinate transformation of (t, x, y, z)→(t, r, θ, ϕ), i. e. ,

$$
\left(\begin{array}{c} ct \\ \hline x \\ y \\ z \end{array}\right)=\left(\begin{array}{c} ct \\ \hline r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\theta \end{array}\right) \qquad \cdots (3)
$$

Corollary 3.

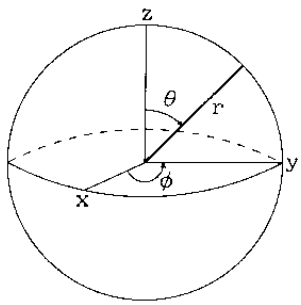
The connection by the force received from the rest mass M_0 with distance r is

$$
\frac{d}{d\tau}\left(\begin{array}{c} dct/d\tau \\ \hline dx/d\tau \\ dy/d\tau \\ dz/d\tau \end{array}\right)=\left\{-\frac{GM_0}{r^2}\left(\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{array}\right)\frac{dct}{d\tau}-\left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)\frac{d\theta}{d\tau}-\left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -\sin\theta \\ 0 & 0 & 0 & -\cos\theta \\ 0 & \sin\theta & \cos\theta & 0 \end{array}\right)\frac{d\phi}{d\tau}\right\}\left(\begin{array}{c} dct/d\tau \\ \hline dr/d\tau \\ rd\theta/d\tau \\ r\sin\theta d\phi/d\tau \end{array}\right)
$$

Proof.

From (3)

$$
\left(\begin{array}{c} dct/d\tau \\ \hline dx/d\tau \\ dy/d\tau \\ dz/d\tau \end{array}\right)=\left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & \sin\theta\cdot\cos\phi & \cos\theta\cdot\cos\phi & -\sin\phi \\ 0 & \sin\theta\cdot\sin\phi & \cos\theta\cdot\sin\phi & \cos\phi \\ 0 & \cos\theta & -\sin\theta & 0 \end{array}\right)\left(\begin{array}{c} dct/d\tau \\ \hline dr/d\tau \\ rd\theta/d\tau \\ r\sin\theta d\phi/d\tau \end{array}\right).
$$



Therefore

$$
\frac{d}{d\tau}\left(\begin{array}{c} dct/d\tau \\ \hline dx/d\tau \\ dy/d\tau \\ dz/d\tau \end{array}\right)=\frac{d}{d\tau}\left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & \sin\theta\cdot\cos\phi & \cos\theta\cdot\cos\phi & -\sin\phi \\ 0 & \sin\theta\cdot\sin\phi & \cos\theta\cdot\sin\phi & \cos\phi \\ 0 & \cos\theta & -\sin\theta & 0 \end{array}\right)\left(\begin{array}{c} dct/d\tau \\ \hline dr/d\tau \\ rd\theta/d\tau \\ r\sin\theta d\phi/d\tau \end{array}\right)
$$

$$
+\left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & \sin\theta\cdot\cos\phi & \cos\theta\cdot\cos\phi & -\sin\phi \\ 0 & \sin\theta\cdot\sin\phi & \cos\theta\cdot\sin\phi & \cos\phi \\ 0 & \cos\theta & -\sin\theta & 0 \end{array}\right)\frac{d}{d\tau}\left(\begin{array}{c} dct/d\tau \\ \hline dr/d\tau \\ rd\theta/d\tau \\ r\sin\theta d\phi/d\tau \end{array}\right),
$$

and then the connection in the polar coordinate is

$$\frac{d}{d\tau}\left(\begin{array}{r} dct/d\tau \\ \hline dr/d\tau \\ rd\theta/d\tau \\ r\sin\theta d\phi/d\tau \end{array}\right) = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & \sin\theta\cdot\cos\phi & \cos\theta\cdot\cos\phi & -\sin\phi \\ 0 & \sin\theta\cdot\sin\phi & \cos\theta\cdot\sin\phi & \cos\phi \\ 0 & \cos\theta & -\sin\theta & 0 \end{array}\right)^{-1}$$

$$\left(-\frac{GM_0}{r^2}\left(\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ \hline \sin\theta\cdot\cos\phi & 0 & i\sin\phi & i\cos\theta\cdot\cos\phi \\ \sin\theta\cdot\sin\phi & 0 & -i\cos\phi & i\cos\theta\cdot\sin\phi \\ \cos\theta & 0 & 0 & -\sin\theta \end{array}\right)\frac{dct}{d\tau} - \frac{d}{d\tau}\left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & \sin\theta\cdot\cos\phi & \cos\theta\cdot\cos\phi & -\sin\phi \\ 0 & \sin\theta\cdot\sin\phi & \cos\theta\cdot\sin\phi & \cos\phi \\ 0 & \cos\theta & -\sin\theta & 0 \end{array}\right)\right)\left(\begin{array}{r} dct/d\tau \\ \hline dr/d\tau \\ rd\theta/d\tau \\ r\sin\theta d\phi/d\tau \end{array}\right)$$

$$= \left(-\frac{GM_0}{r^2}\left(\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{array}\right)\frac{dct}{d\tau} - \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)\frac{d\theta}{d\tau} - \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -\sin\theta \\ 0 & 0 & 0 & -\cos\theta \\ 0 & \sin\theta & \cos\theta & 0 \end{array}\right)\frac{d\phi}{d\tau}\right)\left(\begin{array}{r} dct/d\tau \\ \hline dr/d\tau \\ rd\theta/d\tau \\ r\sin\theta d\phi/d\tau \end{array}\right)$$

q. e. d.

This connection is as follows :

$$d^2ct/d\tau^2 = -mr^{-2}(dct/d\tau)(dr/d\tau) \qquad \cdots (4)$$

$$d^2r/d\tau^2 = -mr^{-2}(dct/d\tau)^2 + r(d\theta/d\tau)^2 + r\sin^2\theta(d\phi/d\tau)^2 \qquad \cdots (5)$$

$$d(rd\theta)/d\tau^2 = -imr^{-1}\sin\theta(dct/d\tau)(d\phi/d\tau) - (d\theta/dt)(dr/d\tau) + r\sin\theta\cdot\cos\theta(d\phi/d\tau)^2 \qquad \cdots (6)$$

$$d(r\sin\theta d\phi)/d\tau^2 = imr^{-1}(dct/d\tau)(d\phi/d\tau) - \sin\theta(d\phi/d\tau)(dr/d\tau) - r\cos\theta(d\phi/d\tau)(d\theta/d\tau) \qquad \cdots (7)$$

where $m = GM_0$

§ 3. Property of solution of differential equations

Let's $\theta = \pi/2 - i\Theta$ then the connection is

$$d^2ct/d\tau^2 = -mr^{-2}(dct/d\tau)(dr/d\tau) \qquad \cdots (4)'$$

$$d^2r/dt^2 = -[mr^{-2} - r^{-1}((rch\Theta d\phi/dct)^2 - (rd\Theta/dct)^2)](dct/d\tau)^2 \qquad \cdots (5)'$$

$$d(r^2d\Theta/d\tau)/d\tau = [mr^{-2} - sh\Theta d\phi/dct](dct/d\tau)(r^2ch\Theta d\phi/d\tau) \qquad \cdots (6)'$$

$$d(r^2ch\Theta d\phi/d\tau)/d\tau = [mr^{-2} - sh\Theta d\phi/dct](dct/d\tau)(r^2d\Theta/d\tau) \qquad \cdots (7)'$$

From (4)' $\div (dct/d\tau)$

$$d\log(dct/d\tau)/d\tau = (d[mr^{-1}]/d\tau)$$

$$c\gamma = dct/d\tau = De^{m/r}, \text{ D is constant} \qquad \cdots (8)$$

From (5)'

$$d^2r/dct^2 = -[mr^{-2} - r^{-1}(rd\phi/dct)^2](dct/d\tau)^2$$

$$\text{where } rd\phi/dct = ((rch\Theta d\phi/dct)^2 - (rd\Theta/dct)^2)^{1/2} \qquad \cdots (9)$$

From (6)'+(7)' and (6)'−(7)'

$$d(r^2 ch\Theta d\phi/d\tau \pm r^2 d\Theta/d\tau)/d\tau = \pm[mr^{-2}dct/d\tau - sh\Theta d\phi/d\tau](r^2 ch\Theta d\phi/d\tau \pm r^2 d\Theta/d\tau)$$

$$dlong(r^2 ch\Theta d\phi/d\tau \pm r^2 d\Theta/d\tau)d\tau = \pm(mr^{-2}dct/d\tau - sh\Theta d\phi/d\tau)$$

$$\log(r^2 ch\Theta d\phi/d\tau \pm r^2 d\Theta/d\tau) = \pm\int(mr^{-2}dct - sh\Theta d\phi)$$

$$r^2 ch\Theta d\phi/d\tau \pm r^2 d\Theta/d\tau = E\exp[\pm\int(mr^{-2}dct - sh\Theta d\phi)],$$ E is a constant.

Therefore

$$r^2 ch\Theta d\phi/d\tau = E ch\int(mr^{-2}dct - sh\Theta d\phi) \quad \cdots (6)''$$

$$r^2 d\Theta/d\tau = E sh\int(mr^{-2}dct - sh\Theta d\phi) \quad \cdots (7)''$$

$$d\Theta/ch\Theta d\phi = th\int(mr^{-2}dct - sh\Theta d\phi).$$

and

From $(6)''^2-(7)''^2$

$$r^2 d\phi/d\tau = ((r^2 ch\Theta d\phi/d\tau)^2-(r^2 d\Theta/d\tau)^2)^{1/2} = E \quad \cdots (10)$$

this is the law of area.

From (8), (9) and (10), an orbit equation holds as follows :

$$d^2(1/r)/d\phi^2 + 1/r = m(D/E)^2 e^{2m/r}$$
$$= m/(r^2 d\phi/dct)^2 \quad \cdots (A)$$

Because from (10)

$$dr/dct = (E/r^2)dr/d\phi$$
$$= -Ed(1/r)/d\phi \quad \cdots (B)$$

and

$$d^2r/dct^2 = -Ed^2(1/r)/d\phi^2(d\phi/dct)$$
$$= -(E/r)^2 d^2(1/r)/d\phi^2 \quad \cdots (C)$$

hold.

the correspondence between a speed at the perihelion r_1 and the orbit of the planet is as follows :

a speed (at the perihelion r_1)	an orbit
$r_1 d\phi_1/dt = (m/r_1)^{1/2}$	a circle
$(m/r_1)^{1/2} < r_1 d\phi_1/dt < (1-e^{-2m/r1})^{1/2}$	a quasi-ellipse
$r_1 d\phi_1/dt = (1-e^{-2m/r1})^{1/2}$	a quasi-parabola
$(1-e^{-2m/r1})^{1/2} < r_1 d\phi_1/dt < \cdots$	a quasi-hyperbola

Because from the equation (A) and (C), a point r_0 such that $d^2r/dct^2 = d^2(1/r)/d\phi^2 = 0$ is

$$r_0 = (E/D)^2 e^{-2m/r0}/m, \quad r_0 d\phi_0/dct = (m/r_0)^{1/2} \quad \cdots (D)$$

This is the same speed as a planet moving on the circle with radius r_0.

Multiply (a) by $2d(1/r)/d\phi$ and integral it, then

$$2d(1/r)/d\phi \cdot d^2(1/r)/d\phi^2 + 2/r \cdot d(1/r)/d\phi = 2m(D/E)^2 e^{2m/r} \cdot d(1/r)/d\phi$$

$$(d(1/r)/d\phi)^2 + 1/r^2 - (D/E)^2 e^{2m/r} = 1/r_1^2 - (D/E)^2 e^{2m/r1}$$
$$= -(c/E)^2$$

where a distance r_1 at the perihelion is the one which satisfies the equation $dr/dct = d(1/r)/d\phi = 0$ and $(De^{m/r})^2 - (E/r)^2 = (dct/d\tau)^2 - (dr/d\tau)^2 - (rd\phi/d\tau)^2 = c^2$

Then

$(D/c)^2 e^{2m/r_1} - (E/cr_1)^2 = 1, \quad r_1 d\phi_1/dct = (1-c/D)^2 e^{\ 2m/r_1})^{1/2}$ … (E)

And a distance r_2 at the aphelion is

$1/r_2^2 - (D/E)^2 e^{2m/r_2} = 1/r_1^2 - (D/E)^2 e^{2m/r_1} = -(c/E)^2$,

$(E/D)^2 = (e^{2m/r_2} - e^{2m/r_1})/(1/r_2^2 - 1/r_1^2) = mr_0 e^{2m/r_0}$ … (F)

When $r_2 \to \infty$ (quasi ellipse) then $(E/D)^2 = r_1^2(e^{2m/r_1} - 1) = e^{2m/r_0}/mr_0$, $r_1 d\phi_1/dt - (1 - e^{-2m/r_1})^{1/2}$ (an escape velocity) and $D = c$.

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