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A 4-dimensional Force and Electromagnetism*

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Abstract

In this paper, we discuss the time component E_t of an electric field derived in a previous paper[4)] and the force caused by it.

Contents:

In § 1 we review a traditional Coulomb's law, Ampere's law and Biot-Savart's law and study their examples.

In § 2 we study the 4-dimensional force which act on the moving electric charge in the electromagnetic field.

In § 3 we find a force induced by the time component E_t of the electricfield.

In § 4 we rewrite the Lorentz transformation by the matrix SL (2, $\mathbb{C}$) in the general case.

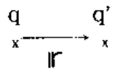
§ 1 Force and field caused by a charge and a current

In this section, we review two traditional laws in electromagnetism.

(Case A-Coulomb's law)

The force $\mathbf{f}$ on an electric charge q' caused by an another electric charge q the distance $\mathbf{r}$ away is given by

$$\mathbf{f} = \frac{q' q}{4\pi\epsilon_0 r^2} \frac{\mathbf{r}}{r} \quad \text{(by experience).}$$



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We usually interpret this force as a result of the electricfield E caused by the charge q ,i,e.,

$$\mathbf{f} = q'E,$$

and

$$E = \frac{q}{4\pi\epsilon_0 r^2} \; \frac{\mathbf{r}}{r} \text{ (newton/coulomb)}$$

where $\epsilon_0 = 8.854 \times 10^{-12}$ (coulomb²/meter²·newton).

Moreover this electricfield E is represented as $E = -\text{grad}\,\phi$ by a scalar potential ϕ of a charge q ,i,e.,

$$\phi = -\int_{\infty}^{r} E\,ds = -\int_{\infty}^{r} \frac{q}{4\pi\epsilon_0 r^2}\,dr = \frac{q}{4\pi\epsilon_0 r}.$$

When the charge density ρ is distributed continuously, the potential is

$$\phi = \frac{1}{4\pi\epsilon_0} \int \frac{\rho}{r}\,dv$$

and the electricfield is

$$E = -\text{grad}\,\phi = \frac{1}{4\pi\epsilon_0} \int \frac{\rho}{r^2} \frac{\mathbf{r}}{r}\,dv.$$

(example 1)

In the case A, when two charges q, q' have a same charge one coulomb and the distance is one meter then the force is

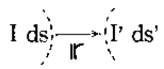
$$\mathbf{f} = \frac{1}{4\pi\epsilon_0} = 9.0 \times 10^9 \text{ (newton)}.$$

This intensity is very strong compared with that of the gravitational force.

(Case B Ampere's law)

The force on a current I' caused by an another current I the distance $\mathbf{r}$ away is given by

$$\mathbf{f} = \frac{\mu_0}{4\pi} \int\int \frac{I'\,ds' \times I\,ds \times \mathbf{r}}{r^3} \quad \text{(by experience)}$$



We usually interpret this force as a result of the magneticfield B induced by the current I ,i.e.,

$$\mathbf{f} = \int I'ds' \times B$$

and

$$B = \frac{\mu_0}{4\pi} \int \frac{I\,ds \times \mathbf{r}}{r^3} \text{ (weber/merter}^2\text{)}$$

where $\mu_0 = 1.257 \times 10^{-6}$ (weber/ampere·meter)

Moreover this magneticfield B is represented as B=rot A by a vector potential A of a current I , i.e.,

$$A = \frac{\mu_0}{4\pi} \int \frac{I\,ds}{r}$$

Specially, a vector potential of an infinitesimal current element I ds is

$$dA = \frac{\mu_0}{4\pi} \frac{I\,ds}{r}$$

and the magnetic field is

$$dB = \frac{\mu_0}{4\pi} \frac{I\,ds \times \mathbf{r}}{r^3} \text{ (Biot-Savart's law).}$$

Therefore a force on an individual moving charge q' which speed is v in a magnetic field dB is

$df = q'v \times dB$ where $q'v = I'\,ds'$.

(example 2)

The action and reaction forces between two infinitesimal current elements are as follows :

The action force $d\mathbf{F}$ is

$$d\mathbf{F} = \frac{\mu_0}{4\pi} \frac{I'\,ds' \times I\,ds \times \mathbf{r}}{r^3}$$

$$= -\frac{\mu_0}{4\pi} \frac{(I'\,ds' \cdot I\,ds)\,\mathbf{r}}{r^3} + \underline{\frac{\mu_0}{4\pi} \frac{(I'\,ds' \cdot \mathbf{r})\,I\,ds}{r^3}}$$

and the reaction force dF' is

$$d\mathbf{F}' = -\frac{\mu_0}{4\pi} \frac{I\,ds \times I'\,ds' \times \mathbf{r}}{r^3}$$

$$= \frac{\mu_0}{4\pi} \frac{(I\,ds \cdot I'\,ds')\,\mathbf{r}}{r^3} - \underline{\frac{\mu_0}{4\pi} \frac{(I\,ds \cdot \mathbf{r})\,I'\,ds'}{r^3}} .$$

Above two force $d\mathbf{F}$ and $-d\mathbf{F}'$ are not the same by the underlined part, this result contradicts the law of action and reaction but when we integrate the force on the whole line,

$$\frac{\mu_0}{4\pi} \int \frac{(I'\,ds' \cdot \mathbf{r})\,I\,ds}{r^3} = \frac{\mu_0\,I\,ds}{4\pi} \int \mathrm{grad}\,\frac{1}{r}\,I'\,ds'$$

$$= 0$$

hold.

We can revive the law of action and reaction in this example by introducing a time component E_t of a electric field in the § 3 .

§ 2 4-dimensional force and the moving charge

We define a 4-dimensional force F as

$$F: \begin{cases} F_t = d^2 m_0 ct/d^2 c\tau \\ F_x = d^2 m_0 x/d^2 c\tau \\ F_y = d^2 m_0 y/d^2 c\tau \\ F_z = d^2 m_0 z/d^2 c\tau \end{cases}$$

where m_0 is a rest mass, τ is a proper time.

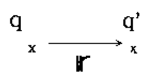
> Proposition 3.
>
> A 4-dimensional force F on a moving electric charge q in the electromagnetic field E and B is as follows :
>
> $F_t = j \cdot E$ (variation rate of energy)
>
> $F = j_0 E + j \times B$ (Lorentz force)
>
> where $j_0 = q\gamma$, $j = q\gamma u/c$ and u is the velocity of charge q.

Proof

The force between two standing charges q, q' is as follows:

The electric field is

$$E = \frac{1}{4\pi\,\epsilon_0} \frac{q}{r^2} \frac{\mathbf{r}}{r}$$



and E induce the force to the electric charge q'

$$F = \frac{q'\,q}{4\pi\epsilon_0 r^2} \frac{\mathbf{r}}{r} = q'\,E$$

as shown in § 1, case A.

We observe this situation on the moving coordinate with speed $-u_x$ in the x-direction, then by the proposition 4 below, the 4-dimensional force F^{-u} and the electromagnetic field E^{-u}, B^{-u} on this coordinate are as follows :

(4-dimensional force)

$$F^{-u}:\quad \begin{array}{l} F_t^{-u}=q'E_x\gamma u_x/c \\ F_x^{-u}=q'E_x\gamma \\ F_y^{-u}=q'E_y \\ F_z^{-u}=q'E_z \end{array}$$

and

(electromagnetic field)

$$E^{-u}:\quad \begin{array}{l} E_x^{-u}=E_x \\ E_y^{-u}=E_y\gamma \\ E_z^{-u}=E_z\gamma \end{array} \qquad B^{-u}:\quad \begin{array}{l} B_x^{-u}=0 \\ B_y^{-u}=-E_z\gamma u_x/c\ . \\ B_z^{-u}=\ E_y\gamma u_x/c \end{array}$$

On the other hand, this force F^{-u} which acts on the moving charge q with speed u_x is caused by the above field E^{-u} and B^{-u}, i.e.,

By the Coulomb's law, the force on the charge part $j_o'=q'r$ in the field E^{-u}, B^{-u} is

$$F(j_o'):\quad \begin{array}{l} F_t(j_o') = 0 \text{ (no variation of energy)} \\ F_x(j_o') = j_o'\cdot E_x \\ F_y(j_o') = j_o'\cdot E_y\gamma \\ F_z(j_o') = j_o'\cdot E_z\gamma \end{array} \qquad (j_o, j_x) \xrightarrow[\ r\ ]{} (j_o', j_x')$$

and by the Ampere's law, the force on the current part $j_x'=q'ru_x/c$ in the field E^{-u}, B^{-u} is

$$F(j_x'):\quad \begin{array}{l} F_t(j_x') = \text{(unknown quantity)} \\ F_x(j_x') = 0 \\ F_y(j_x') = -j_x'\cdot E_y j_x \\ F_z(j_x') = -j_x'\cdot E_z j_x\ . \end{array}$$

Therefore we compare the force F^u with the sum $F(j_o') + F(j_x')$ of two forces on a charge part j_o' and a current part j_x' then

$$F_t(j_x') = q'E_x\gamma u_x/c = j_x'E_x\ .$$

In general, when a charge moves for an arbitrary direction

$F_t(j') = j'\cdot E$ where " $\cdot$ " is a scalar product

holds. q.e.d.

Proposition 4.

When a coordinate move with pneed u_x for x-direction, a 4-dimensional force and electromagnetic field are transformed by Lorentz transformation as follows :

(a 4-dimensional force)

$$F^u:\quad \begin{array}{l} F_t^u = \gamma(F_t - u_x/c\cdot F_x) \\ F_x^u = \gamma(F_x - u_x/c\cdot F_t) \\ F_y^u = F_y \\ F_z^u = F_z \end{array}$$

and

(a electromagnetic field)

$$E^u - iB^u:\quad \begin{array}{l} E_x^u - iB_x^u = E_x - iB_x \\ E_y^u - iB_y^u = \gamma(E_y - iB_y) - i\gamma u_x/c\cdot(E_z - iB_z) \\ E_z^u - iB_z^u = \gamma(E_z - iB_z) + i\gamma u_x/c\cdot(E_y - iB_y) \end{array}$$

More general case and its representation by the matrix $SL(2, \mathbb{C})$ is discussed in § 4.

§ 3 A force induced by the time-component E_t of a elecric field

We consider the situation where the charge density $\rho(x,y,z)$ and charge velocity $u(x,y,z)$ at each point are time independent ,i.e., stationary current: $\partial\rho/\partial t=0$, div $I=0$.

definition 5.

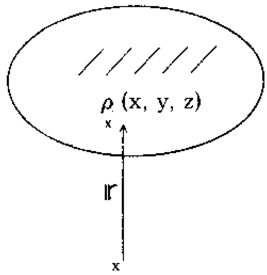
A electric charge and current is a 4-dimensional vector, and therefore the electric charge and current density is as follows:

$$\rho(x,y,z) = \rho_0\, \gamma \text{ and } I(x,y,z) = \rho_0\, \gamma u/c$$
$$= \rho u/c$$

where ρ_0 is a rest charge and c is a velocity of light.

We define a potential and a field as follows:

A scalar potential is



$$\phi=\frac{1}{4\pi\epsilon_0}\int \frac{\rho}{r}\, dv,$$

and it induces the electric field E

$$=\frac{1}{4\pi\epsilon_0}\int \frac{\rho}{r^2}\frac{\mathbf{r}}{r}\, dv.$$

A vector potential is

$$A=\frac{1}{4\pi\epsilon_0}\int \frac{I}{r}\, dv \text{ (correspond to cA in § 1, case B)}$$

and it induces magnetic field B and a new field E_t which is a time component of a electric field.

$$B=\text{rot } A$$
$$=\frac{1}{4\pi\epsilon_0}\int \frac{I}{r^2}\times\frac{\mathbf{r}}{r}\, dv \text{ (correspond to cB in § 1, case B)}$$
$$E_t=\text{div } A$$
$$=\frac{1}{4\pi\epsilon_0}\int \frac{I}{r^2}\cdot\frac{\mathbf{r}}{r}\, dv$$

where "×" and "·" mean a vector product and a scalar product respectively.

Theorem 6.

The 4-dimensional force on a moving charge q with speed u in the field E_t, E, B is

$$F_t = \underline{j_0 E_t} + j\cdot E$$
$$F = j_0 E + \underline{j E_t} + j\times B$$

where the underlined parts are new terms and $j_0 = q\gamma$, $j = q\gamma u/c$.

And this force is a real part of a complex force $(\tilde{F}_t, \tilde{F})$ as

$$\begin{cases}\tilde{F}_t = j_0 E_t + j\cdot E - \underline{i\, j\cdot B} \\ \tilde{F} = j_0 E + j E_t + j\times B - \underline{i(j_0 B - j\times E)}\end{cases}$$

where the underlined parts are imaginary ones.

and this force is represented by a matrix form as follows :

$$\begin{pmatrix}\tilde{F}_t+\tilde{F}_x & \tilde{F}_y+i\tilde{F}_z \\ \tilde{F}_y-i\tilde{F}_z & \tilde{F}_t-\tilde{F}_x\end{pmatrix} = \begin{pmatrix}E_t+(E_x-iB_x) & (E_y-iB_y)+i(E_z-B_z) \\ (E_y-iB_y)-i(E_z-B_z) & E_t-(E_x-iB_x)\end{pmatrix}\begin{pmatrix}j_0+j_x & j_y+ij_z \\ j_y-ij_z & j_0-j_x\end{pmatrix}$$

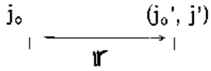
Proof

We calculate a 4-dimensional action and reaction force between a infinitesimal current element (j_o, j) and (j_o', j')

(Case A) the force between a charge part j_o of (j_o, j) and a infinitesimal current element (j_o', j').

The field induced by a charge part j_o is

$$E=\frac{1}{4\pi\epsilon_0}\frac{j_o}{r^2}\frac{\mathbf{r}}{r}$$



where $j_o=q\gamma$

$j_o'=q'\gamma'$ and $j'=q'\gamma' u'/c$

Therefore the force on (j_o', j') is

$F_t=j'\cdot E$

$F=j_o'E.$

On the other hand, a charge part j_o receives the reaction force $(-F_o, -F)$ and this force is represented by a field E', B', E_t' as follows :

$-F_t=j_oE_t'$

$-F=j_oE'$

where

$$E'=-\frac{1}{4\pi\epsilon_0}\frac{j_o'}{r^2}\frac{\mathbf{r}}{r},\quad B'=-\frac{1}{4\pi\epsilon_0}\frac{j'}{r^2}\times\frac{\mathbf{r}}{r},\quad E_t'=-\frac{1}{4\pi\epsilon_0}\frac{j'}{r^2}\cdot\frac{\mathbf{r}}{r}$$

is induced by a infinitesimal current element (j_o', j').

Because the reaction force on a charge part j_o from a field E', B', E_t' is

$$\begin{aligned}-F_t&=-j'\cdot E\\&=-j'\cdot\left(\frac{1}{4\pi\epsilon_0}\frac{j_o}{r^2}\frac{\mathbf{r}}{r}\right)\\&=j_o\left(-\frac{1}{4\pi\epsilon_0}\frac{j'}{r^2}\cdot\frac{\mathbf{r}}{r}\right)\\&=j_oE_t'\end{aligned}$$

$$\begin{aligned}-F&=-j_o'E\\&=-j_o'\left(\frac{1}{4\pi\epsilon_0}\frac{j_o}{r^2}\frac{\mathbf{r}}{r}\right)\\&=j_o\left(-\frac{1}{4\pi\epsilon_0}\frac{j_o'}{r^2}\frac{\mathbf{r}}{r}\right)\\&=j_oE'\end{aligned}$$

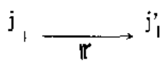
(Case B) the force between a infinitesimal current element (j_o, j) and charge part j_o'.

We get a same result as case A

(Case C) the force between two current parts j oa (j_o, j) and j' of (j_o', j').

The fields induced by a current part j are

$$B=\frac{1}{4\pi\epsilon_0}\frac{j}{r^2}\times\frac{\mathbf{r}}{r},$$



and

where $j=q\gamma u/c$ and $j'=q'\gamma' u'/c$

$$E_t = \frac{1}{4\pi\epsilon_0}\ \frac{j}{r^2}\cdot\frac{\mathbf{r}}{r}\ ,$$

Therefore the force on j' received from a field B, E_t is

$F_t = 0$ (no variation of energy)

$F = F(B) + F(E_t)$

and F(B) is the force from B

$$\begin{aligned} F(B) &= j'\times B \\ &= -j'\times\left(\frac{1}{4\pi\epsilon_0}\ \frac{j}{r^2}\times\frac{\mathbf{r}}{r}\right) \\ &= \left(j\cdot\frac{1}{4\pi\epsilon_0}\ \frac{j'}{r^2}\right)\frac{\mathbf{r}}{r} - j\cdot\left(\frac{1}{4\pi\epsilon_0}\ \frac{j'}{r^2}\cdot\frac{\mathbf{r}}{r}\right) \\ &= \underline{\left(j\cdot\frac{1}{4\pi\epsilon_0}\ \frac{j'}{r^2}\right)\frac{\mathbf{r}}{r}} + j\cdot E_0'\ , \end{aligned}$$

and $F(E_t)$ is the force from E_t

$F(E_t) =$ (unknown quantity).

On the other hand, a current j is received the reaction force $(-F_0, -F)$ from fields B', E_t' induced by a current part j', i.e.,

$$B' = -\frac{1}{4\pi\epsilon_0}\ \frac{j'}{r^2}\times\frac{\mathbf{r}}{r} \quad \text{and} \quad E_t' = -\frac{1}{4\pi\epsilon_0}\ \frac{j'}{r^2}\cdot\frac{\mathbf{r}}{r}\ ,$$

then

$-F_t = 0$

$-F = F(B') + F(E_t')$

and F(B') is the force from B'

$$\begin{aligned} F(B') &= j\times B' \\ &= -\underline{\left(j'\cdot\frac{1}{4\pi\epsilon_0}\ \frac{j}{r^2}\right)\frac{\mathbf{r}}{r}} - j'\cdot E_t \end{aligned}$$

and $F(E_t')$ is the force from E_t'

$F(E_t') =$ (unknown quantity).

Therefore we compar an action force $F(B)+F(E_o)$ with a reaction force $F(B')+F(E_o')$, then we get

$F(E_t) = j'\cdot E_t$ and $F(E_t') = j\cdot E_t'$

because the underlined parts are the same.

This means that j' in a field B, E_t is received the force $F(j') = j'\times B + j'E_t$.

q.e.d.

(example 7.)

We calculate the action and reaction forces between two infinitesimal current elements as follows :

The fields induced by the element (j_o, j) are

$$E = \frac{1}{4\pi\epsilon_0}\ \frac{j_o}{r^2}\ \frac{\mathbf{r}}{r}\ , \qquad (j_o, j)\ |\xrightarrow[\mathbf{r}]{}|\ (j_o', j')$$

$$B = \frac{1}{4\pi\epsilon_0}\ \frac{j}{r^2}\times\frac{\mathbf{r}}{r} \quad \text{and} \quad E_0 = \frac{1}{4\pi\epsilon_0}\ \frac{j}{r^2}\cdot\frac{\mathbf{r}}{r}\ ,$$

Therefore the action force is

$F_o = j_o' E_t + j'\cdot E$

$$=j_o' \left(\frac{1}{4\pi\epsilon_o}\frac{j}{r^2}\cdot\frac{\mathbb{r}}{r}\right) + j'\cdot\left(\frac{1}{4\pi\epsilon_o}\frac{j_o}{r^2}\frac{\mathbb{r}}{r}\right),$$

$$F = j_o'E + j'E_t + j'\times B$$

$$=j_o' \left(\frac{1}{4\pi\epsilon_o}\frac{j_o}{r^2}\frac{\mathbb{r}}{r}\right) + j' \left(\frac{1}{4\pi\epsilon_o}\frac{j}{r^2}\cdot\frac{\mathbb{r}}{r}\right) + j'\times\left(\frac{1}{4\pi\epsilon_o}\frac{j}{r^2}\times\frac{\mathbb{r}}{r}\right)$$

$$=\frac{1}{4\pi\epsilon_o}\frac{j_o' j_o}{r^2}\frac{\mathbb{r}}{r} + j\left(\frac{1}{4\pi\epsilon_o}\frac{j'}{r^2}\cdot\frac{\mathbb{r}}{r}\right) - \frac{1}{4\pi\epsilon_o}\frac{j'\cdot j}{r^2}\frac{\mathbb{r}}{r} + \left(\frac{1}{4\pi\epsilon_o}\frac{j}{r^2}\cdot\frac{\mathbb{r}}{r}\right) j'.$$

And the reaction force is

$$F_o' = j_o\, E_t' + j\cdot E'$$

$$= -j_o \left(\frac{1}{4\pi\epsilon_o}\frac{j'}{r^2}\cdot\frac{\mathbb{r}}{r}\right) - j\cdot\left(\frac{1}{4\pi\epsilon_o}\frac{j_o'}{r^2}\frac{\mathbb{r}}{r}\right),$$

$$F' = j_oE' + jE_t' + j\times B'$$

$$= -\frac{1}{4\pi\epsilon_o}\frac{j_oj_o'}{r^2}\frac{\mathbb{r}}{r} - j'\left(\frac{1}{4\pi\epsilon_o}\frac{j}{r^2}\cdot\frac{\mathbb{r}}{r}\right) + \frac{1}{4\pi\epsilon_o}\frac{j\cdot j'}{r^2}\frac{\mathbb{r}}{r} - \left(\frac{1}{4\pi\epsilon_o}\frac{j'}{r^2}\frac{\mathbb{r}}{r}\right) j.$$

Therefore the reaction force is an opposed direction of action one.

§ 4 A general form of the Lorentz transformation and a its matrix form

A Lorentz transformation to a moving coordinate with speed $u = u_x$ for the x-direction is

$$\begin{cases} ct' = \gamma\,(ct-\beta x) \\ x' = \gamma\,(x-\beta ct) \\ y', z' = y, z \qquad \text{where } \beta = u_x/c \end{cases}$$

We calculate a Lorentz transformation in a general case , i.e., the direction cosine of moving coordinate is (A, B, C).

Then

$$\begin{cases} ct' = \gamma ct - \gamma\beta Ax - \gamma\beta By - \gamma\beta Cz \\ x' = -\gamma\beta Act + (\gamma A^2 + B^2 + C^2)x + (\gamma AD + BE + CF)y + (\gamma AG + BH + CI)z \\ y' = -\gamma\beta Dct + (\gamma DA + EB + FC)x + (\gamma D^2 + E^2 + F^2)y + (\gamma DG + EH + FI)z \\ z' = -\gamma\beta Gct + (\gamma GF + HB + IC)x + (\gamma GD + HE + IF)y + (\gamma G^2 + H^2 + I^2)z \end{cases}$$

where $A^2+B^2+C^2=1,\quad D^2+E^2+F^2=1,\quad G^2+H^2+I^2=1$

$AD+BE+CF=0,\quad AG+BH+CI=0,\quad DG+EH+FI=0$

Proposition 8.

The Lorentz translation to the moving coordinate with direction cosine (A, B, C) is

$$\begin{aligned} ct' &= \gamma ct - \gamma\beta Ax - \gamma\beta By - \gamma\beta Cz \\ x' &= -\gamma\beta Act + (1+(\gamma-1)A^2)x + (\gamma-1)ABy + (\gamma-1)CAz \\ y' &= -\gamma\beta Bct + (\gamma-1)ABx + (1+(\gamma-1)B^2)y + (\gamma-1)BCz \\ z' &= -\gamma\beta Cct + (\gamma-1)ACx + (\gamma-1)BCy + (1+(\gamma-1)C^2)z \end{aligned}$$

and its matrix form is

$$\begin{pmatrix} ct'+x' & y'+iz' \\ y'-iz' & ct'-x' \end{pmatrix} = \begin{pmatrix} \gamma_+ - \gamma_- A & -\gamma_-(B+iC) \\ -\gamma_-(B-iC) & \gamma_+ + \gamma_- A \end{pmatrix} \begin{pmatrix} ct+x & y+iz \\ y-iz & ct-x \end{pmatrix} \begin{pmatrix} \gamma_+ - \gamma_- A & -\gamma_-(B+iC) \\ -\gamma_-(B-iC) & \gamma_+ + \gamma_- A \end{pmatrix}$$

where

$$\gamma_+ = ((\gamma+1)/2)^{1/2},\ \gamma_- = ((\gamma-1)/2)^{1/2}$$

Proof

Let

$$\begin{pmatrix} ct'+x' & y'+iz' \\ y-iz' & ct'-x' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} ct+x & y+iz \\ y-iz & ct-x \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix},$$

and we compare the components in the both side.

then

$$\begin{cases} 2ct'=(|a|^2+|b|^2+|c|^2+|d|^2)\, ct+(|a|^2-|b|^2+|c|^2-|d|^2)\, x \\ \qquad +(b\bar{a}+a\bar{b}+d\bar{c}+c\bar{d})y-i(b\bar{a}-a\bar{b}+d\bar{c}-c\bar{d})z \\ 2x'=(|a|^2+|b|^2-|c|^2-|d|^2)\, ct+(|a|^2-|b|^2-|c|^2+|d|^2)\, x \\ \qquad +(b\bar{a}+a\bar{b}-d\bar{c}-c\bar{d})\, y-iIb\bar{a}-a\bar{b}-d\bar{c}+c\bar{d})\, z \\ 2y'=(a\bar{c}+b\bar{d}+c\bar{a}+d\bar{b})\, ct+(a\bar{c}'-b\bar{d}'+c\bar{a}'-d\bar{b})\, x \\ \qquad +(b\bar{c}+a\bar{d}+d\bar{a}+c\bar{a})\, y-i\, (b\bar{c}-a\bar{d}+d\bar{a}-c\bar{a})\, z \\ 2iz'=(a\bar{c}+b\bar{d}-c\bar{a}-d\bar{b})ct+(a\bar{c}-b\bar{d}-c\bar{a}+d\bar{b})\, x \\ \qquad +(b\bar{c}+a\bar{d}-d\bar{a}-c\bar{a})\, y-i\, (b\bar{c}-a\bar{d}-d\bar{a}+c\bar{a})\, z \end{cases}$$

Therefore we decided the components a, b, c, d, as follows :

$$\begin{cases} |a|^2+|b|^2+|c|^2+|d|^2=2\gamma \\ |a|^2-|b|^2+|c|^2-|d|^2=-2\gamma\beta A \\ b\bar{a}+a\bar{b}+d\bar{c}+c\bar{d} = -2\gamma\beta B \\ b\bar{a}-a\bar{b}-d\bar{c}-c\bar{d} = -2i\gamma\beta C \\ \\ |a|^2+|b|^2-|c|^2-|d|^2=-2\gamma\beta A \\ |a|^2-|b|^2-|c|^2+|d|^2=2(\gamma A^2+B^2+C^2) \\ b\bar{a}+a\bar{b}-d\bar{c}-c\bar{d} =2(\gamma AD+BE+CF) \\ b\bar{a}-a\bar{b}-d\bar{c}+c\bar{d} =2i(\gamma AG+BH+CI) \\ \\ a\bar{c}+b\bar{d}+c\bar{a}+d\bar{b}=-2\gamma\beta D \\ a\bar{c}-b\bar{d}+c\bar{a}-d\bar{b}=2(\gamma DA+EB+FC) \\ b\bar{c}+a\bar{d}+d\bar{a}+c\bar{a}=2(\gamma D^2+E^2+F^2) \\ b\bar{c}-a\bar{d}+d\bar{a}-c\bar{a}=2i(\gamma DG+EH+FI) \\ \\ a\bar{c}+b\bar{d}-c\bar{a}-d\bar{b}=-2\gamma\beta G \\ a\bar{c}-b\bar{d}-c\bar{a}+d\bar{b}=2(\gamma GA+HB+IC) \\ b\bar{c}+a\bar{d}-d\bar{a}-c\bar{a}=2(\gamma GD+HE+IF) \\ b\bar{c}-a\bar{d}-d\bar{a}+c\bar{a}=2i(\gamma G^2+H^2+I^2) \end{cases}$$

then

$$\begin{cases} 2|a|^2=\gamma-2\gamma\beta A+(\gamma A^2+B^2+C^2) \\ 2|b|^2=\gamma-(\gamma A^2+B^2+C^2) \\ 2|c|^2=\gamma-(\gamma A^2+B^2+C^2) \\ 2|d|^2=\gamma+2\gamma\beta A+(\gamma A^2+B^2+C^2) \end{cases}$$

Specially, when the components a, d is real and $b = \bar{c}$,

$$\begin{cases} a=((\gamma+1)/2)^{1/2}-((\gamma-1)/2)^{1/2}A \\ b=-((\gamma-1)/2)^{1/2}(B+iC) \\ c=-((\gamma-1)/2)^{1/2}(B-iC) \\ d=((\gamma+1)/2)^{1/2}+((\gamma-1)/2)^{1/2}A \end{cases}$$

hold. q.e.d.

Corollary 9.[4)]

Let $\gamma_+=((\gamma+1)/2)^{1/2}$, $\gamma_-=((\gamma-1)/2)^{1/2}$,then the Lorentz transformation of the differential matrix is

$$\begin{pmatrix} \partial/\partial ct'+\partial/\partial x' & \partial/\partial y'+i\partial/\partial z' \\ \partial/\partial y'-i\partial/\partial z' & \partial/\partial ct'-\partial/\partial x' \end{pmatrix}$$
$$=\begin{pmatrix} \gamma_++\gamma_-A & \gamma_-(B+iC) \\ \gamma_-(B-iC) & \gamma_+-\gamma_-A \end{pmatrix} \begin{pmatrix} \partial/\partial ct'+\partial/\partial x' & \partial/\partial y'+i\partial/\partial z' \\ \partial/\partial y'-i\partial/\partial z' & \partial/\partial ct'-\partial/\partial x' \end{pmatrix} \begin{pmatrix} \gamma_++\gamma_-A & \gamma_-(B+iC) \\ \gamma_-(B-iC) & \gamma_+-\gamma_-A \end{pmatrix}.$$

And Lorentz transformation of the charge matrix is

$$\begin{pmatrix} j_o'+j_x' & j_y'+ij_z \\ j_y'-ij_z' & j_o'-j_x' \end{pmatrix}$$
$$=\begin{pmatrix} \gamma_+-\gamma_-A & -\gamma_-(B+iC) \\ -\gamma_-(B-iC) & \gamma_++\gamma_-A \end{pmatrix} \begin{pmatrix} j_o'+j_x' & j_y'+ij_z' \\ j_y'-ij_z' & j_o'-j_x' \end{pmatrix} \begin{pmatrix} \gamma_+-\gamma_-A & -\gamma_-(B+iC) \\ -\gamma_-(B-iC) & \gamma_++\gamma_-A \end{pmatrix}.$$

And Lorentz transformation of potential matrix is

$$\begin{pmatrix} 1/c\cdot\phi'+A_x' & A_y'+iA_z' \\ A_y'-iA_z' & 1/c\cdot\phi'-A_x' \end{pmatrix}$$
$$=\begin{pmatrix} \gamma_+-\gamma_-A & -\gamma_-(B+iC) \\ -\gamma_-(B-iC) & \gamma_++\gamma-A \end{pmatrix} \begin{pmatrix} 1/c\cdot\phi'+A_x' & A_y'+iA_z' \\ A_y'-iA_z' & 1/c\cdot\phi'-A_x' \end{pmatrix} \begin{pmatrix} \gamma_+-\gamma_-A & -\gamma_-(B+iC) \\ -\gamma_-(B-iC) & \gamma_++\gamma_-A \end{pmatrix}.$$

And Lorentz transformation of the field matrix and components are

$$\begin{pmatrix} (E_t'-iB_t')+(E_x'-iB_x') & (E_y'-iB_y')+i(E_z'-iB_z') \\ (E_y'-iB_y')-i(E_z'-iB_z') & (E_t'-iB_t')-(E_x'-iB_x') \end{pmatrix}$$
$$=\begin{pmatrix} \gamma_+-\gamma_-A & -\gamma-(B+iC) \\ -\gamma_-(B-iC) & \gamma_++\gamma_-A \end{pmatrix} \begin{pmatrix} (E_t-iB_t)+(E_x-iB_x) & (E_y-iB_y)+i(E_z-iB_z) \\ (E_y-iB_y)-i(E_z-iB_z) & (E_t-iB_t)-(E_x-iB_x) \end{pmatrix} \begin{pmatrix} \gamma_++\gamma_-A & \gamma_-(B+iC) \\ \gamma_-(B-iC) & \gamma_+-\gamma_-A \end{pmatrix}$$

,i.e.,

$$\begin{cases} \mathbf{E}_t'=\mathbf{E}_t \\ \mathbf{E}_x'=(\gamma-(\gamma-1)A^2)\mathbf{E}_x-((\gamma-1)AB+i\gamma\beta C)\mathbf{E}_y-((\gamma-1)AC+i\gamma\beta B)\mathbf{E}_z \\ \mathbf{E}_y'=-((\gamma-1)AB-i\gamma\beta C)\mathbf{E}_x+(\gamma-(\gamma-1)B^2)\mathbf{E}_y-((\gamma-1)BC+i\gamma\beta A)\mathbf{E}_z \\ \mathbf{E}_z'=-((\gamma-1)AC-i\gamma\beta B)\mathbf{E}_x-((\gamma-1)BC)-i\gamma\beta A)\mathbf{E}_y+(\gamma-(\gamma-1)C^2)\mathbf{E}_z \end{cases}$$

where $\mathbf{E}_t=E_t+iB_t$, $\mathbf{E}_x=E_x+iB_x$, $\mathbf{E}_y=E_y+iB_y$, $\mathbf{E}_z=E_z+iB_z$.

And Lorentz transformation of the force matrix and components are

$$\begin{pmatrix} F_t'+F_x' & F_y'+iF_z' \\ F_y'-iF_z' & F_t'-F_x' \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_+ - \gamma_- A & -\gamma_-(B+iC) \\ -\gamma_-(B-iC) & \gamma_+ + \gamma_- A \end{pmatrix} \begin{pmatrix} F_t' + F_x' & F_y' + iF_z' \\ F_y' - iF_z' & F_t' - F_x' \end{pmatrix} \begin{pmatrix} \gamma_+ - \gamma_- A & -\gamma_-(B+iC) \\ -\gamma_-(B-iC) & \gamma_+ + \gamma - A \end{pmatrix}$$

,i.e.,

$$\begin{cases} F_t' = \gamma F_t - \gamma\beta A F_x - \gamma\beta B F_y - \gamma\beta C F_z \\ F_x' = -\gamma\beta A F_t + (1+(\gamma-1)A^2)F_x + (\gamma - 1)ABF_y + (\gamma - 1)CAF_z \\ F_y' = -\gamma\beta B F_t + (\gamma - 1)ABF_x + (1 + (\gamma - 1)B^2)F_y - (\gamma - 1)BCF_z \\ F_z' = -\gamma\beta C F_t + (\gamma - 1)ACF_x - (\gamma - 1)BCF_y + (1 + (\gamma - 1)C^2)F_z \end{cases}.$$

Proof

We define the electromagnetic field of matrix type as

$$\begin{pmatrix} (E_t - iB_t) + (E_x - iB_x) & (E_y - iB_y) + i(E_z - iB_z) \\ (E_y - iB_y) - i(E_z - iB_z) & (E_t - iB_t) - (E_x - iB_x) \end{pmatrix}$$

$$= \begin{pmatrix} \partial/\partial ct - \partial/\partial x & -\partial/\partial y - i\partial/\partial z \\ -\partial/\partial y + i\partial/\partial z & \partial/\partial ct + \partial/\partial x \end{pmatrix} \begin{pmatrix} 1/c\cdot\phi - A_x & -A_y - iA_z \\ -A_y + iA_z & 1/c\cdot\phi + A_x \end{pmatrix}$$

then

$$\begin{pmatrix} (E_t' - iB_t') + (E_x' - iB_x') & (E_y' - iB_y') + i(E_z' - iB_z') \\ (E_y' - iB_y') - i(E_z' - iB_z') & (E_t' - iB_t') - (E_x' - iB_x') \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_+ - \gamma_- A & -\gamma_-(B+iC) \\ -\gamma_-(B-iC) & \gamma_+ + \gamma_- A \end{pmatrix} \begin{pmatrix} \partial/\partial ct - \partial/\partial x & -\partial/\partial y - i\partial/\partial z \\ -\partial/\partial y + i\partial/\partial z & \partial/\partial ct + \partial/\partial x \end{pmatrix} \begin{pmatrix} \gamma_+ - \gamma_- A & -\gamma_-(B+iC) \\ -\gamma_-(B-iC) & \gamma_+ + \gamma_- A \end{pmatrix}$$

$$\cdot \begin{pmatrix} \gamma_+ + \gamma_- A & \gamma_-(B+iC) \\ \gamma_-(B-ic) & \gamma_+ - \gamma_- A \end{pmatrix} \begin{pmatrix} 1/c\cdot\phi - A_x & -A_y - iA_z \\ -A_y + iA_z & 1/c\cdot\phi + A_x \end{pmatrix} \begin{pmatrix} \gamma_+ + \gamma_- A & \gamma_-(B+iC) \\ \gamma_-(B-iC) & \gamma_+ - \gamma_- A \end{pmatrix},$$

$$= \begin{pmatrix} \gamma_+ - \gamma_- A & -\gamma_-(B+iC) \\ -\gamma_-(B-iC) & \gamma_+ + \gamma_- A \end{pmatrix} \begin{pmatrix} (E_t - iB_t) + (E_x - iB_x) & (E_y - iB_y) + i(E_z - iB_z) \\ (E_y - iB_y) - i(E_z - iB_z) & (E_t - iB_t) - (E_x - iB_x) \end{pmatrix} \begin{pmatrix} \gamma_+ + \gamma_- A & \gamma_-(B+iC) \\ \gamma_-(B-iC) & \gamma_+ - \gamma_- A \end{pmatrix}.$$

We define the 4-dimensional force of matrix type as

$$\begin{pmatrix} F_o + F_x & F_y + iF_z \\ F_y - iA_z & F_o - F_x \end{pmatrix}$$

$$= \begin{pmatrix} (E_t - iB_t) + (E_x - iB_x) & (E_y - iB_y) + i(E_z - iB_z) \\ (E_y - iB_y) - i(E_z - iB_z) & (E_t - iB_t) - (E_x - iB_x) \end{pmatrix} \begin{pmatrix} j_o + j_x & j_y + ij_z \\ j_y - ij_z & j_o - j_x \end{pmatrix},$$

then

$$\begin{pmatrix} F_o' + F_x' & F_y' + iF_z' \\ F_y' - iF_z' & F_o' - F_x' \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_+ - \gamma_- A & -\gamma_-(B+iC) \\ -\gamma_-(B-iC) & \gamma_+ + \gamma_- A \end{pmatrix} \begin{pmatrix} (E_t - iB_t) + (E_x - iB_x) & (E_y - iB_y) + i(E_z - iB_z) \\ (E_y - iB_y) - i(E_z - iB_z) & (E_t - iB_t) - (E_x - iB_x) \end{pmatrix} \begin{pmatrix} \gamma_+ + \gamma_- A & \gamma_-(B+iC) \\ \gamma_-(B-iC) & \gamma_+ - \gamma_- A \end{pmatrix}$$

$$\cdot\begin{pmatrix}\gamma_+-\gamma_-A & -\gamma_-(B+iC)\\ -\gamma_-(B-iC) & \gamma_++\gamma_-A\end{pmatrix}\begin{pmatrix}j_o+j_x & j_x+ij_z\\ j_y-ij_z & j_o-j_x\end{pmatrix}\begin{pmatrix}\gamma_+-\gamma_-A & -\gamma_-(B+iC)\\ -\gamma_-(B-iC) & \gamma_++\gamma_-A\end{pmatrix}$$

$$=\begin{pmatrix}\gamma_+-\gamma_-A & -\gamma_-(B+iC)\\ -\gamma_-(B-iC) & \gamma_++\gamma_-A\end{pmatrix}\begin{pmatrix}F_o'+F_x' & F_y'+iF_z'\\ F_y'-iF_z' & F_o'-F_x'\end{pmatrix}\begin{pmatrix}\gamma_+-\gamma_-A & -\gamma_-(B+iC)\\ -\gamma_-(B-iC) & \gamma_++\gamma_-A\end{pmatrix}.$$

q.e.d.

References

(1) M. F. Atiyah N. J. Hitchin: CONSTRUCTION OF INSTANTONS, Phys. Lett. 65A-3 (1978) 185-187.
(2) S. Helgason: Differential Geometry, Lie Groups, and Symmetric Spaces Academic Press, Inc. (1978)
(3) Kobayashi and Nomizu: Foundations of Differential Geometry I Interscience Publishers (1969)
(4) Y. Takemoto: An extension of Maxwell's equations and the deduction from a Yang-Mills functional Bull. of NBU Vol. 19, No. 2 (1991-Oct.)
(5) Y. Takemoto: "Pseudo"-Fiber Bundle and Connection on it Bull. of NBU Vol. 20, No. 1 (1992-Feb.)
(6) Y. Takemoto: The Transformation Group on an Extended Hopf Fiber Bundle and Its Associated Bundle Bull. of NBU Vol. 20, No. 2 (1992-Oct.)
(7) Y. Takemoto: A Gauge Theory on the Anti-de Sitter Space Bull. of NBU Vol. 21, No. 1 (1993-Feb.)