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A Gauge Theory on the Anti-de Sitter Space*

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Abstract

We discuss the gauge theory on anti-de Sitter space which has the Levi-Civita connection.

In §1 we review and study a gauge transformation group on which preserves the Levi-Civita connection.

In §2 we define the gauge field (i.e., geuge potential and field strength) and derivative whose field has the space-component and a new (time-component) factor.

In §3 we lead the equation which is satisfied by the gauge field (c.f. Yang-Mills' theory)

In §4 we apply the theorem in the proceeding section to the gauge group sl$(2, \mathbb{C})$, and lead the approximate form of Newton's equation of motion under the universal gravitation from the gravitational potential.

§1. Introduction

We consider an extended Hoph fiber bundle[5)]

$$\begin{tikzcd}
 & S^7(\mathbb{C}) = \left\{ \begin{pmatrix} X \\ Y \end{pmatrix} \in M_2(\mathbb{C})^2 / |X|+|Y|=1 \right\} \arrow[d, "\pi"] \\
{}^4 \arrow[ur, "\mu_{\pm 1}"] \arrow[r] & S^4(\mathbb{C}) = S^7(\mathbb{C})/SL(2, \mathbb{C})
\end{tikzcd}$$

where $|X|$, $|Y|$ are determinant of X and Y.

and another fiber bundle[6)]which is constructed from view point of the group theory.
And we also call the latter an extended Hoph fiber bundle ,i.e.,

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$$
\begin{tikzcd}
 & Sp'(2,\mathbb{C}) \arrow[d, "\pi"] \\
\mathbb{C}^4 \arrow[ur, "\mu_{+1}"] \arrow[r] & S^4(\mathbb{C}) = Sp'(2,\mathbb{C}) / \begin{pmatrix} SL(2,\mathbb{C}) & 0 \\ 0 & SL(2,\mathbb{C}) \end{pmatrix}
\end{tikzcd}
\quad \overset{\rho}{\cong} \text{(locally)} \quad
\begin{tikzcd}
SO(2,3,\mathbb{C}) \arrow[d] \\
SO(2,3,\mathbb{C})/SO(1,3,\mathbb{C})
\end{tikzcd}
$$

(extended Hoph fiber bundle) (associated frame bundle)

where the fiber space $Sp'(2,\mathbb{C})$ is

$$
Sp'(2,\mathbb{C}) = \left\{ g = \begin{pmatrix} a\ b \\ c\ d \end{pmatrix} \in SL(4,\mathbb{C}) / \tilde{g}\cdot g = E \right\}, \qquad \tilde{g} = \begin{pmatrix} a\ c \\ \tilde{b}\ \tilde{d} \end{pmatrix}
$$

$$
= \left\{ g = \begin{pmatrix} a\ b \\ c\ d \end{pmatrix} \in SL(4,\mathbb{C}) / \begin{matrix} \tilde{a}b + \tilde{c}d = 0 \\ |a| + |c| = 1, |b| + |d| = 1 \end{matrix} \right\}
$$

and the projection is

$$
\pi : \begin{pmatrix} a\ b \\ c\ d \end{pmatrix} \to u = bd^{-1} (= -\tilde{a}^{-1}\tilde{c}) \qquad \in \mathbb{C}^4(|u| \neq -1)
$$

or

$$
\to u' = \tilde{b}^{-1}\tilde{d} (= -ac^{-1}) = \tilde{u}^{-1} \in \mathbb{C}^4(|u'| \neq -1)
$$

and the structure group is $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$.

Morerover the cross sections on $\mathbb{C}^4(|u| \neq -1)$ and $\mathbb{C}^4(|u'| \neq -1)$ are

$$
\mathbb{C}^4(|u| \neq -1) \ni u \to \mu'_{-1}(u) = \begin{pmatrix} (1+|u|)^{-1/2}E & (1+|u|)^{-1/2}u \\ -(1+|u|)^{-1/2}\tilde{u} & (1+|u|)^{-1/2}E \end{pmatrix},
$$

$$
\mathbb{C}^4(|u'| \neq -1) \ni u' \to \mu'_{1}(u') = \begin{pmatrix} -(1+|u'|)^{-1/2}u' & (1+|u'|)^{-1/2}E \\ (1+|u'|)^{-1/2}E & (1+|u'|)^{-1/2}\tilde{u}' \end{pmatrix}.
$$

And the canonical connection A_{-1} on $\mathbb{C}^4(|u| \neq -1)$, A_1 on $\mathbb{C}^4(|u'| \neq -1)$ are as follows:
Let

$$
g(u) = \mu'_{-1}(u) = \frac{1}{(1+|u|)^{1/2}} \begin{pmatrix} E & u \\ -\tilde{u} & E \end{pmatrix}
$$

then

$$
A = g^{-1}dg = \frac{1}{1+|u|} \begin{pmatrix} \mathrm{Im}\, ud\tilde{u} & du \\ -d\tilde{u} & \mathrm{Im}\, \tilde{u}du \end{pmatrix}.
$$

Therefore

$$
A_{-1} = \frac{1}{1+|u|} \begin{pmatrix} \mathrm{Im}\, ud\tilde{u} & 0 \\ 0 & \mathrm{Im}\, \tilde{u}du \end{pmatrix}
$$

and for the $g(u') = \mu_1(u')$, the connection is

$$
A_1 = \frac{1}{1+|u'|} \begin{pmatrix} \mathrm{Im}\, \tilde{u}'du' & 0 \\ 0 & \mathrm{Im}\, u'd\tilde{u} \end{pmatrix}.
$$

Moreover these connections correspond to Levi-Civita connection on the associated frame bundle by the following locally isomorphism[6)]

$$\rho: Sp'(2,\mathbb{C}) \rightarrow SO(2,3,\mathbb{C})$$

and

$$SL(2,\mathbb{C})\times SL(2,\mathbb{C}) \rightarrow SO(1,3,\mathbb{C}).$$

The transformation group $Sp'(2,\mathbb{C})$ which is called a gauge group on the above extended Hopf fiber bundle preserve the canonical connection and therefore we consider the product bundle $P=S^4(\mathbb{C})\times Sp'(2,\mathbb{C})$ and its adjoint bundle $sp'(2,\mathbb{C})_P = P\times_{Ad} sp'(2,\mathbb{C})$ on $S^4(\mathbb{C})$ whose fiber is the Lie algebra $sp'(2,\mathbb{C})$ of the Lie group $Sp'(2,\mathbb{C})$, i.e.,

$$sp'(2,\mathbb{C}) = \left\{ X=\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in sl(4,\mathbb{C}) / \tilde{X}+X=0 \right\}$$

$$= \left\{ X=\begin{pmatrix} A & B \\ -\tilde{B} & D \end{pmatrix} \in sl(4,\mathbb{C}) / A, D \in sl(2,\mathbb{C}) \right\}$$

and this Lie algebra corresponds to the Lie algebra $so(2,3,\mathbb{C})$ of Lie group $SO(2,3,\mathbb{C})$.

Definition 1.

we define a conjugation "=" and a real part of $SL(4,\mathbb{C})$ as follows :

Let

$$\begin{array}{l} SL_R(4,\mathbb{C}) \\ (Sp^*(2,\mathbb{C})^{6)}) \end{array} = \left\{ g=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(4,\mathbb{C}) / \begin{array}{c} {}^t\bar{g}\cdot J_4\cdot g=J_4 \\ \text{(real equation)} \end{array} \right\},\ J_4=\begin{pmatrix} 0 & E_2 \\ -E_2 & 0 \end{pmatrix}$$

$$= \left\{ g=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(4,\mathbb{C}) / \begin{array}{c} {}^ta\bar{c}={}^tc\bar{a},\ {}^tb\bar{d}={}^td\bar{b} \\ {}^ta\bar{d}-{}^tc\bar{b}=E_2 \end{array} \right\} \begin{array}{c} {}^t(ac^{-1})=a\bar{c}^{-1},\ {}^t(bd^{-1})=b\bar{d}^{-1} \\ \text{or} \\ {}^t(ca^{-1})=c\bar{a}^{-1},\ {}^t(db^{-1})=d\bar{b}^{-1} \end{array}$$

be a Lie subgroup of $SL(4,\mathbb{C})$ and its Lie algebra is

$$\begin{array}{l} sl_R(4,\mathbb{C}) \\ (sp^*(2,\mathbb{C})^{6)}) \end{array} = \left\{ X=\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in sl(4,\mathbb{C}) / \begin{array}{c} {}^t\bar{X}\cdot J_4+J_4\cdot X=0 \\ \text{(real equation)} \end{array} \right\}$$

$$= \left\{ X=\begin{pmatrix} A & B \\ C & -{}^t\bar{A} \end{pmatrix} \in sl(4,\mathbb{C}) / A \in sl(2,\mathbb{C}),\ B, C \text{ :hermitian} \right\}$$

these Lie subgroup and Lie subalgebra correspond to the $SO(2,4)$ and $so(2,4)$ respectively under ρ.

Let the conjugation "=" be

$$\overline{\overline{\begin{pmatrix} A & B \\ C & D \end{pmatrix}}} = \begin{pmatrix} -{}^t\bar{D} & {}^t\bar{B} \\ {}^t\bar{C} & -{}^t\bar{A} \end{pmatrix} \quad \text{on } sl(4,\mathbb{C})$$

then

$$\overline{\overline{Re}}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{1}{2}\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} + \overline{\overline{\begin{pmatrix} A & B \\ C & D \end{pmatrix}}} \right) = \begin{pmatrix} \frac{1}{2}(A-{}^t\bar{D}) & \frac{1}{2}(B+{}^t\bar{B}) \\ \frac{1}{2}(C+{}^t\bar{C}) & \frac{1}{2}(D-{}^t\bar{A}) \end{pmatrix}$$

$$\overline{\overline{Im}}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{1}{2}\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} - \overline{\overline{\begin{pmatrix} A & B \\ C & D \end{pmatrix}}} \right) = \begin{pmatrix} \frac{1}{2}(A+{}^t\bar{D}) & \frac{1}{2}(B-{}^t\bar{B}) \\ \frac{1}{2}(C-{}^t\bar{C}) & \frac{1}{2}(D+{}^t\bar{A}) \end{pmatrix}$$

therefore the following relation holds

$$sl(4,\mathbb{C}) = sl_R(4,\mathbb{C}) + i\cdot sl_R(4,\mathbb{C})$$

and this conjugation corresponds the usual one on the Lie algebra $SO(2,4,\mathbb{C})$.

We restrict $SL(4,\mathbb{C})$ and $sl(4,\mathbb{C})$ to the Lie subgroup $Sp'(2,\mathbb{C})$ and Lie subalgebra $sp'(2,\mathbb{C})$

Let

$$\begin{array}{ll} Sp_R'(2,\mathbb{C}) & = Sp'(2,\mathbb{C}) \cap SL_R(4,\mathbb{C}) \\ (Sp^*{}_0(2,\mathbb{C})^{6)}) & \qquad (Sp^*(2,\mathbb{C})) \\ & = \left\{ g = \begin{pmatrix} a & b \\ -{}^t\tilde{b} & {}^t\tilde{a} \end{pmatrix} \in SL(4,\mathbb{C})/\tilde{g}\cdot g = E \right\} \end{array}$$

and

$$\begin{array}{ll} sp_R'(2,\mathbb{C}) & = sp'(2,\mathbb{C}) \cap sl_R(4,\mathbb{C}) \\ (sp^*{}_0(2,\mathbb{C})^{6)}) & \qquad (sp^*(2,\mathbb{C})) \\ & = \left\{ X = \begin{pmatrix} A & B \\ -\tilde{B} & -{}^t\bar{A} \end{pmatrix} \in sL(4,\mathbb{C})/A \in sl(2,\mathbb{C}),\ b \text{ hermitian} \right\} \end{array}$$

These Lie subgroup and Lie subalgebra correspond to the SO(2,3) and so(2,3) respectively under ρ then the conjugation "$=$" is

$$\overline{\overline{\begin{pmatrix} A & B \\ -\tilde{B} & D \end{pmatrix}}} = \begin{pmatrix} -{}^t\bar{D} & {}^t\bar{B} \\ -{}^t\widetilde{\overline{B}} & -{}^t\bar{A} \end{pmatrix} \quad \text{on } sp'(2,\mathbb{C})$$

and

$$\overline{\overline{\mathrm{Re}}} \begin{pmatrix} A & B \\ -\tilde{B} & D \end{pmatrix} = \frac{1}{2} \left(\begin{pmatrix} A & B \\ -\tilde{B} & D \end{pmatrix} + \overline{\overline{\sigma}} \begin{pmatrix} A & B \\ -\tilde{B} & D \end{pmatrix} \right) = \begin{pmatrix} \frac{1}{2}(A-{}^t\bar{D}) & \frac{1}{2}(B+{}^t\bar{B}) \\ \frac{1}{2}(-\tilde{B}-{}^t\widetilde{\overline{B}}) & \frac{1}{2}(D-{}^t\bar{A}) \end{pmatrix}$$

$$\overline{\overline{\mathrm{Im}}} \begin{pmatrix} A & B \\ -\tilde{B} & D \end{pmatrix} = \frac{1}{2} \left(\begin{pmatrix} A & B \\ -\tilde{B} & D \end{pmatrix} - \overline{\overline{\sigma}} \begin{pmatrix} A & B \\ -\tilde{B} & D \end{pmatrix} \right) = \begin{pmatrix} \frac{1}{2}(A+{}^t\bar{D}) & \frac{1}{2}(B-{}^t\bar{B}) \\ \frac{1}{2}(-\tilde{B}+{}^t\widetilde{\overline{B}}) & \frac{1}{2}(D+{}^t\bar{A}) \end{pmatrix}$$

therefore the following relation holds

$$sp(2,\mathbb{C}) = sp'_R(2,\mathbb{C}) + i\cdot sp'_R(2,\mathbb{C}).$$

§2 The additional and corrected point concerning to the curvature tensor

Example 1. (the case of electromagnetic field)

Let $A = \phi dt + A_x dx + A_y dy + A_z dz \in u(1)$ be a connection from (i.e., electromagnetic potential) on the Minkowski space, then the curvature of A is as follows :

Three components of the electric field are

$$E_x = -\partial_t A_x - \partial_x \phi,\quad E_y = -\partial_t A_y - \partial_y \phi,\quad E_z = -\partial_t A_z - \partial_z \phi$$

and three components of the magnetic field are

$$B_x = \partial_z A_y - \partial_y A_z,\ B_y = \partial_x A_z - \partial_z A_x,\ B_z = \partial_y A_x - \partial_x A_y$$

and we define a new factor[4] (i.e., a time-component of the electric and magnetic field) as

$$E_t = \underline{\partial_t \phi + \partial_x A_x + \partial_y A_y + \partial_z A_z}$$

$$B_t = 0.$$

Then we define a potential matix , a derivative matrix and a curvature matrix as follows :

$$A = \begin{pmatrix} A_1 + A_2 & A_3 + iA_4 \\ A_3 - iA_4 & A_1 - A_2 \end{pmatrix} \quad , \quad D = \begin{pmatrix} \partial/\partial x^1 + \partial/\partial x^2 & \partial/\partial x^3 + i\partial/\partial x^4 \\ \partial/\partial x^3 - i\partial/\partial x^4 & \partial/\partial x^1 - \partial/\partial x^2 \end{pmatrix}$$

$$F=\tilde{D}\tilde{A}$$
$$=\begin{pmatrix} E_t+iB_t+(E_x+iB_x) & E_y+iB_y+i(E_z+iB_z) \\ E_y+iB_y-i(E_z+iB_z) & E_t+iB_t-(E_x+iB_x) \end{pmatrix}$$

where each $\tilde{A}$ and $\tilde{D}$ is a cofactor matrix of A and D respectively.

In following example 2, example 3, $\tilde{A}$ is a cofactor matrix of A, when we look upon the components A_1, A_2, A_3, A_4 (matrix) as a scalar

and each $\widetilde{\mathrm{Re}}\,A=\frac{1}{2}(A+\tilde{A})$,$\widetilde{\mathrm{Im}}\,A=\frac{1}{2}(A-\tilde{A})$ is a real part and imaginary part of A respectively with respect to the conjugation "=".

Example 2. (the case of $SL(2,\mathbb{C})$)

Let $A=\sum_{i=1}^{4} A_i(x)dx^i \in sl(2,\mathbb{C})$ (where $x^1=t, x^2=x, x^3=y, x^4=z$) be a connection form.

then the curvature form is

$$F(A)=dA+\frac{1}{2}[A\wedge A]$$
$$=\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{4} F_{ij}(x)\,dx^i\wedge dx^j,\quad F_{ij}(x)=-F_{ji}(x)$$

where $\quad F_{ij}(x)=\partial_i A_j(x)-\partial_j A_i(x)+[A_i(x),A_j(x)] \qquad (i,j=1\sim4)$.

Let

$$E_x=-F_{12},\ E_y=-F_{13},\ E_z=-F_{14}$$
$$B_x=F_{34},\ B_y=F_{42},\ B_z=F_{23}$$

and a new factor (a time-component of the curvature) is

$$E_t=\partial_1A_1(x)+\partial_2A_2(x)+\partial_3A_3(x)+\partial_4A_4(x)$$
$$B_t=0.$$

Then the curvature matrix is

$$F=\tilde{D}\tilde{A}+\frac{1}{2}[A\cdot\tilde{A}]$$
$$=\begin{pmatrix} \partial_1-\partial_2 & -\partial_3-i\partial_4 \\ \partial_3+i\partial_4 & \partial_1+\partial_2 \end{pmatrix}\begin{pmatrix} A_1-A_2 & -A_3-iA_4 \\ -A_3+iA_4 & A_1+A_2 \end{pmatrix}+\frac{1}{2}\left(\begin{pmatrix} A_1+A_2 & A_3+iA_4 \\ A_3-iA_4 & A_1-A_2 \end{pmatrix}\cdot\begin{pmatrix} A_1-A_2 & -A_3-iA_4 \\ -A_3+iA_4 & A_1+A_2 \end{pmatrix}\right)$$

where $[A\cdot B]$ is means that the matrix product $A\cdot B$ and the bracket product between the compoment is done at the same time

,i.e.,

$$[A\cdot\tilde{A}]=\begin{pmatrix} \begin{matrix}[A_1,A_1]-[A_2,A_2]-[A_3,A_3]-[A_4,A_4]\\ -[A_1,A_2]+[A_2,A_1]+i[A_3A_4]-i[A_4,A_3]\end{matrix} & \begin{matrix}-[A_1,A_3]-i[A_2,A_4]+[A_3,A_1]+i[A_4,A_2]\\ -i[A_1,A_4]-[A_2,A_3]+[A_3,A_2]+i[A_4,A_1]\end{matrix} \\ \begin{matrix}-[A_1,A_3]-i[A_2,A_4]+[A_3,A_1]+i[A_4,A_2]\\ +i[A_1,A_4]+[A_2,A_3]-[A_3,A_2]-i[A_4,A_1]\end{matrix} & \begin{matrix}[A_1,A_1]-[A_2,A_2]-[A_3,A_3]-[A_4,A_4]\\ +[A_1,A_2]-[A_2,A_1]-i[A_3,A_4]+i[A_4,A_3]\end{matrix} \end{pmatrix}$$

$$=2\begin{pmatrix} -[A_1,A_2]+i[A_3,A_4] & \begin{matrix}-[A_1,A_3]+i[A_4,A_2]\\ +i(-[A_1,A_4]+i[A_2,A_3])\end{matrix} \\ \begin{matrix}-[A_1,A_3]+i[A_4,A_2]\\ +i(-[A_1,A_4]+i[A_2,A_3])\end{matrix} & -[A_1,A_2]+i[A_3,A_4] \end{pmatrix}$$

In Example 1 and 2, we use the conventional curvature and new factor. but in following Example 3, we adopt the new definition involved the conjugate "=".

Example 3. (the case of the gauge group $SP'(2,\mathbb{C})$)

Let $A=\sum_{i=1}^{4} A_i(x)dx^i \in sp'(2,\mathbb{C})$ (where $x^1=t, x^2=x, x^3=y, x^4=z$) be a connection form,

then the <u>curvature matrix</u> is defined by

$$F=\tilde{D}\tilde{A}+\tfrac{1}{2}\,[\overline{\overline{A}}\cdot\tilde{A}]$$

$$=\begin{pmatrix} \partial_1-\partial_2 & -\partial_3-i\partial_4 \\ -\partial_3+i\partial_4 & \partial_1+\partial_2 \end{pmatrix}\begin{pmatrix} A_1-A_2 & -A_3-iA_4 \\ -A_3+iA_4 & A_1+A_2 \end{pmatrix}+\frac{1}{2}\left(\begin{pmatrix} \overline{\overline{A}}_1+\overline{\overline{A}}_2 & \overline{\overline{A}}_3+i\,\overline{\overline{A}}_4 \\ \overline{\overline{A}}_3-i\,\overline{\overline{A}}_4 & \overline{\overline{A}}_1-\overline{\overline{A}}_2 \end{pmatrix}\cdot\begin{pmatrix} A_1-A_2 & -A_3-iA_4 \\ -A_3+iA_4 & A_1+A_2 \end{pmatrix}\right)$$

where

$$[\overline{\overline{A}}\cdot\tilde{A}]=\begin{pmatrix} \begin{array}{c}[\overline{\overline{A}}_1,A_1]-[\overline{\overline{A}}_2,A_2]-[\overline{\overline{A}}_3,A_3]-[\overline{\overline{A}}_4,A_4] \\ -[\overline{\overline{A}}_1,A_2]+[\overline{\overline{A}}_2,A_1]+i[\overline{\overline{A}}_3A_4]-i[\overline{\overline{A}}_4,A_3]\end{array} & \begin{array}{c}-[\overline{\overline{A}}_1,A_3]-i[\overline{\overline{A}}_2,A_4]+[\overline{\overline{A}}_3,A_1]+i[\overline{\overline{A}}_4,A_2] \\ -i[\overline{\overline{A}}_1,A_4]-[\overline{\overline{A}}_2,A_3]+[\overline{\overline{A}}_3,A_2]+i[\overline{\overline{A}}_4,A_1]\end{array} \\ \begin{array}{c}-[\overline{\overline{A}}_1,A_3]-i[\overline{\overline{A}}_2,A_4]+[\overline{\overline{A}}_3,A_1]+i[\overline{\overline{A}}_4,A_2] \\ +i[\overline{\overline{A}}_1,A_4]+[\overline{\overline{A}}_2,A_3]-[\overline{\overline{A}}_3,A_2]-i[\overline{\overline{A}}_4,A_1]\end{array} & \begin{array}{c}[\overline{\overline{A}}_1,A_1]-[\overline{\overline{A}}_2,A_2]-[\overline{\overline{A}}_3,A_3]-[\overline{\overline{A}}_4,A_4] \\ +[\overline{\overline{A}}_1,A_2]-[\overline{\overline{A}}_2,A_1]-i[\overline{\overline{A}}_3,A_4]+i[\overline{\overline{A}}_4,A_3]\end{array} \end{pmatrix}$$

<u>Definition 2.</u>

Let

A, A' be a connection matrix.

We define the action of the covariant derivative $\tilde{D}+A$ on the affine space of connection as follows :

$$(\tilde{D}+A)\tilde{A}'=(\tilde{D}+A)(\tilde{A}+\tilde{A}'-\tilde{A})$$
$$=(\tilde{D}+A)\tilde{A}+(\tilde{D}+A)(\tilde{A}'-\tilde{A})$$
$$=(\tilde{D}+A)\tilde{A}+(\tilde{D}+A)B$$

where $B=\tilde{A}'-\tilde{A}$ is a vector matrix (4-dimensional vector)

and

$$(\tilde{D}+A)\tilde{A}=\tilde{D}\tilde{A}+\tfrac{1}{2}\,[\overline{\overline{A}}\cdot\tilde{A}]=F(A)$$

$$(\tilde{D}+A)\tilde{B}=\tilde{D}\tilde{B}+\ \ [\overline{\overline{A}}\cdot\tilde{B}].$$

Let A_t be a connection matrix such that $A_0=A$.

then

$$F_t=(\tilde{D}+A)\tilde{A}_t$$
$$=(\tilde{D}+A)(\tilde{A}+\tilde{A}_t-\tilde{A})$$
$$=(\tilde{D}+A)\tilde{A}+(\tilde{D}+A)(\tilde{A}_t-\tilde{A})$$
$$=(\tilde{D}\tilde{A}+\tfrac{1}{2}\,[\overline{\overline{A}}\cdot\tilde{A}])+\tilde{D}(\tilde{A}_t-\tilde{A})+\ [\overline{\overline{A}}\cdot\tilde{A}_t-\tilde{A}]$$
$$=(\tilde{D}\tilde{A}_t+\ [\overline{\overline{A}}\cdot A_t])-\tfrac{1}{2}\,[\overline{\overline{A}}\cdot\tilde{A}]$$

and

$$\left.\frac{d}{dt}\right|_{t=0} F_t=\tilde{D}\tilde{\alpha}+\ [\overline{\overline{A}}\cdot\tilde{\alpha}],\text{where } \alpha=\left.\frac{d}{dt}\right|_{t=0} A_t$$

therefore

$$\frac{d}{dt}\Big|_{t=0}(\tilde{F}_t F_t) = \tilde{F}(\tilde{D}\tilde{\alpha} + [\bar{\bar{A}}\cdot\tilde{\alpha}]) + (\tilde{D}\tilde{\alpha} + [\bar{\bar{A}}\cdot\tilde{\alpha}])F.$$

§ 3 The equation of the gauge group

Let $YM(A) = \int tr(\tilde{F}F)\, dv$ be a Yang-Mills functional, then the following theorem holds.

<u>Theorem 3.</u>

Let $\alpha = \dfrac{d}{dt}\Big|_{t=0} A_t$, and we assume that α has a compact support.

then

$$\frac{d}{dt}\Big|_{t=0} YM(A_t) = -\int_M Tr\,(\alpha\cdot DF(A) + DF(A)\cdot\alpha)dv$$
$$+\int_M Tr([\bar{\bar{A}}\cdot\tilde{\alpha}]\cdot\tilde{F}(A) + \tilde{F}(A)\cdot[\bar{\bar{A}}\cdot\tilde{\alpha}])dv.$$

proof

$$\frac{d}{dt}\Big|_{t=0} YM(A_t)$$
$$= \frac{d}{dt}\Big|_{t=0}\int_M Tr\,(\tilde{F}_t\cdot F_t)dv$$
$$= \int_M Tr((\widetilde{\tilde{D}\tilde{\alpha} + \bar{\bar{A}}\cdot\tilde{\alpha}]})\cdot F(A) + \tilde{F}(A)\cdot(\tilde{D}\tilde{\alpha} + [\bar{\bar{A}}\cdot\tilde{\alpha}]))dv$$
$$= Tr\int_M \widetilde{Re}((\tilde{D}\tilde{\alpha} + [\bar{\bar{A}}\cdot\tilde{\alpha}])\cdot\tilde{F}(A) + \tilde{F}(A)\cdot(\tilde{D}\tilde{\alpha} + [\bar{\bar{A}}\cdot\tilde{\alpha}]))dv$$
$$= -Tr\int_M \widetilde{Re}(\alpha\cdot DF(A) + DF(A)\cdot\alpha)dv$$
$$+Tr\int_M \widetilde{Re}([\bar{\bar{A}}\cdot\tilde{\alpha}]\cdot\tilde{F}(A) + \tilde{F}(A)\cdot[\bar{\bar{A}}\cdot\tilde{\alpha}])dv$$
$$= -\int_M Tr(\alpha\cdot DF(A) + DF(A)\cdot\alpha)dv$$
$$+\int_M Tr([\bar{\bar{A}}\cdot\tilde{\alpha}]\cdot\tilde{F}(A) + \tilde{F}(A)\cdot[\bar{\bar{A}}\cdot\tilde{\alpha}])dv$$

q.e.d.

<u>Corollary 4.</u>

$DF(A) + [\tilde{\bar{\bar{A}}}\cdot F(A)] = 0$ is the condition of the Yang-Mills' connection proof

Let α, DF, A (in Theorem 3) be

$$\alpha=\begin{pmatrix}\alpha_t+\alpha_x & \alpha_y+i\alpha_z\\ \alpha_y-i\alpha_z & \alpha_t-\alpha_x\end{pmatrix}\quad DF=\begin{pmatrix}DF_t+DF_x & DF_y+iDF_z\\ DF_y-iDF_z & DF_t-DF_x\end{pmatrix}\quad A=\begin{pmatrix}A_t+A_x & A_y+iA_z\\ A_y-iA_z & A_t-A_x\end{pmatrix}$$

where the component $\alpha_t, \alpha_x, \alpha_y, \alpha_z, DF_t, DF_x, DF_y, DF_z, A_t, A_x, A_y, A_z \in sp'(2, \mathbb{C})$.

Then the first term in Theorem 3 is

$$\begin{aligned}&\alpha\cdot DF(A)\ +\ DF(A)\cdot\alpha\\
&=\begin{pmatrix}\alpha_t+\alpha_x & \alpha_y+i\alpha_z\\ \alpha_y-i\alpha_z & \alpha_t-\alpha_x\end{pmatrix}\begin{pmatrix}DF_t+DF_x & DF_y+iDF_z\\ DF_y-iDF_z & DF_t-DF_x\end{pmatrix}\\
&\quad+\begin{pmatrix}DF_t+DF_x & DF_y+iDF_z\\ DF_y-iDF_z & DF_t-DF_x\end{pmatrix}\begin{pmatrix}\alpha_t+\alpha_x & \alpha_y+i\alpha_z\\ \alpha_y-i\alpha_z & \alpha_t-\alpha_x\end{pmatrix}\end{aligned}$$

$$\begin{aligned}\therefore\ &Tr(\alpha\cdot DF(A)\ +\ DF(A)\cdot\alpha)\\
&=2\ Tr\{(\alpha_t\cdot DF_t+DF_t\cdot\alpha_t)+(\alpha_x\cdot DF_x+DF_x\cdot\alpha_x)+(\alpha_y\cdot DF_y+DF_y\cdot\alpha_y)+(\alpha_z\cdot DF_z+DF_z\cdot\alpha_z)\}.\end{aligned}$$

And the second term in Theorem 3 is

$$\begin{aligned}&[\bar{\bar{A}}\cdot\tilde{\alpha}]\\
&=\begin{pmatrix}\begin{array}{l}[\bar{\bar{A}}_t,\alpha_t]-[\bar{\bar{A}}_x,\alpha_x]-[\bar{\bar{A}}_y,\alpha_y]-[\bar{\bar{A}}_z,\alpha_z]\\ -[\bar{\bar{A}}_t,\alpha_x]+[\bar{\bar{A}}_x,\alpha_t]+i[\bar{\bar{A}}_y,\alpha_z]-i[\bar{\bar{A}}_z,\alpha_y]\end{array} & \begin{array}{l}-[\bar{\bar{A}}_t,\alpha_y]-i[\bar{\bar{A}}_x,\alpha_z]+[\bar{\bar{A}}_y,\alpha_t]+i[\bar{\bar{A}}_z,\alpha_x]\\ -i[\bar{\bar{A}}_t,\alpha_z]-[\bar{\bar{A}}_x,\alpha_y]+[\bar{\bar{A}}_y,\alpha_x]+i[\bar{\bar{A}}_z,\alpha_t]\end{array}\\[2ex]
\begin{array}{l}-[\bar{\bar{A}}_t,\alpha_y]-i[\bar{\bar{A}}_x,\alpha_z]+[\bar{\bar{A}}_y,\alpha_t]+i[\bar{\bar{A}}_z,\alpha_x]\\ +i[\bar{\bar{A}}_t,\alpha_z]+[\bar{\bar{A}}_x,\alpha_y]-[\bar{\bar{A}}_y,\alpha_x]-i[\bar{\bar{A}}_z,\alpha_t]\end{array} & \begin{array}{l}[\bar{\bar{A}}_t,\alpha_t]-[\bar{\bar{A}}_x,\alpha_x]-[\bar{\bar{A}}_y,\alpha_y]-[\bar{\bar{A}}_z,\alpha_z]\\ +[\bar{\bar{A}}_t,\alpha_x]-[\bar{\bar{A}}_x,\alpha_t]-i[\bar{\bar{A}}_y,\alpha_z]+i[\bar{\bar{A}}_z,\alpha_y]\end{array}\end{pmatrix}\end{aligned}$$

therefore

$$\begin{aligned}&Tr([\bar{\bar{A}}\cdot\tilde{\alpha}]\ \tilde{F}(A)+\tilde{F}(A)\ [\bar{\bar{A}}\cdot\tilde{\alpha}])\\
&=2\ Tr\{([\bar{\bar{A}}_t,\alpha_t]-[\bar{\bar{A}}_x,\alpha_x]-[\bar{\bar{A}}_y,\alpha_y]-[\bar{\bar{A}}_z,\alpha_z])\cdot F_t\\
&\qquad+F_t\cdot([\bar{\bar{A}}_t,\alpha_t]-[\bar{\bar{A}}_x,\alpha_x]-[\bar{\bar{A}}_y,\alpha_y]-[\bar{\bar{A}}_z,\alpha_z])\\
&\quad+(-[\bar{\bar{A}}_t,\alpha_x]+[\bar{\bar{A}}_x,\alpha_t]+i[\bar{\bar{A}}_y,\alpha_z]-i[\bar{\bar{A}}_z,\alpha_y])\cdot(-F_x)\\
&\quad+(-F_x)\cdot(-[\bar{\bar{A}}_t,\alpha_x]+[\bar{\bar{A}}_x,\alpha_t]+i[\bar{\bar{A}}_y,\alpha_z]-i[\bar{\bar{A}}_z,\alpha_y])\\
&\quad+(-[\bar{\bar{A}}_t,\alpha_y]-i[\bar{\bar{A}}_x,\alpha_z]+[\bar{\bar{A}}_y,\alpha_t]+i[\bar{\bar{A}}_z,\alpha_x])\cdot(-F_y)\\
&\quad+(-F_y)\cdot(-[\bar{\bar{A}}_t,\alpha_y]-i[\bar{\bar{A}}_x,\alpha_z]+[\bar{\bar{A}}_y,\alpha_t]+i[\bar{\bar{A}}_z,\alpha_x])\\
&\quad+(-i[\bar{\bar{A}}_t,\alpha_z]-[\bar{\bar{A}}_x,\alpha_y]+[\bar{\bar{A}}_y,\alpha_x]+i[\bar{\bar{A}}_z,\alpha_t])\cdot(iF_z)\\
&\quad+(-iF_z)\cdot(i[\bar{\bar{A}}_t,\alpha_z]+[\bar{\bar{A}}_x,\alpha_y]-[\bar{\bar{A}}_y,\alpha_x]-i[\bar{\bar{A}}_z,\alpha_t])\}.\end{aligned}$$

The coefficient of α_t (i.e., $\alpha_x=\alpha_y=\alpha_z=0$) is as follows:

$$\begin{aligned}&2\ Tr(-\alpha_t DF_t-DF_t\alpha_t\\
&\quad+[\bar{\bar{A}}_t,\alpha_t]F_t+F_t[\bar{\bar{A}}_t,\alpha_t]-[\bar{\bar{A}}_x,\alpha_t]F_x-F_x[\bar{\bar{A}}_x,\alpha_t]-[\bar{\bar{A}}_y,\alpha_t]F_y-F_y[\bar{\bar{A}}_y,\alpha_t]-[\bar{\bar{A}}_z,\alpha_t]F_z-F_z[\bar{\bar{A}}_z,\alpha_t])\end{aligned}$$

And we calculate the coefficients of following $\alpha_{t2}\sim\alpha_{t8}$ and $\alpha'_{t2}\sim\alpha'_{t4}$ as follows :

Let

$$\alpha_t=\left(\begin{array}{cc|cc} \alpha_{t2} & \alpha_{t3}+i\alpha_{t4} & \alpha_{t5}+\alpha_{t6} \ \ \alpha_{t7} & \alpha_{t8} \\ \alpha_{t3}-i\alpha_{t4} & -\alpha_{t2} & \alpha_{t7}-i\alpha_{t8} & \alpha_{t5}-\alpha_{t6} \\ \hline -\alpha_{t5}+\alpha_{t6} & \alpha_{t7}+i\alpha_{t8} & \alpha_{t2}' & \alpha_{t3}'+i\alpha_{t4}' \\ \alpha_{t7}-i\alpha_{t8} & -\alpha_{t5}-\alpha_{t6} & \alpha_{t3}'-i\alpha_{t4}' & -\alpha_{t2}' \end{array}\right)$$

and

$$DF_t=\left(\begin{array}{cc|cc} DF_{t2} & DF_{t3}+DF_{t4} & DF_{t5}+DF_{t6} & DF_{t7}+iDF_{t8} \\ DF_{t3}-iDF_{t4} & -DF_{t2} & DF_{t7}-iDF_{t8} & DF_{t5}-DF_{t6} \\ \hline -DF_{t5}+DF_{t6} & DF_{t7}+iDF_{t8} & DF_{t2}' & DF_{t3}'+iDF_{t4}' \\ DF_{t7}-iDF_{t8} & -DF_{t5}-DF_{t6} & DF_{t3}'-iDF_{t4}' & -DF_{t2}' \end{array}\right).$$

then

$$\alpha_t DF_t+DF_t\alpha_t$$

$$\left(\begin{array}{c|c} \begin{array}{l} 2(\alpha_{t2}DF_{t2}+\alpha_{t3}DF_{t3}+\alpha_{t4}DF_{t4} \\ \quad \alpha_{t5}DF_{t5}+\alpha_{t6}DF_{t6}+\alpha_{t7}DF_{t7}+\alpha_{t8}DF_{t8})E \end{array} & \begin{array}{l} (\alpha_{t2}DF_{t6}+\alpha_{t3}DF_{t7}+\alpha_{t4}DF_{t8} \\ +\alpha_{t6}DF_{t2}'+\alpha_{t7}DF_{t3}'+\alpha_{t8}DF_{t4}' \\ \quad +DF_{t2}\alpha_{t6}+DF_{t3}\alpha_{t7}+DF_{t4}\alpha_{t8} \\ +DF_{t6}\alpha_{t2}'+DF_{t7}\alpha_{t3}'+DF_{t8}\alpha_{t4}')E \end{array} \\ \hline \begin{array}{l} (\alpha_{t6}DF_{t2}+\alpha_{t7}DF_{t3}+\alpha_{t8}DF_{t4} \\ +\alpha_{t2}'DF_{t6}+\alpha_{t3}'DF_{t7}+\alpha_{t4}'DF_{t8} \\ \ +DF_{t6}\alpha_{t2}+DF_{t7}\alpha_{t3}+DF_{t8}\alpha_{t4} \\ \ +DF_{t2}'\alpha_{t6}+DF_{t3}'\alpha_{t7}+DF_{t4}'\alpha_{t8})E \end{array} & \begin{array}{l} 2(-\alpha_{t5}DF_{t5}+\alpha_{t6}DF_{t6}+\alpha_{t7}DF_{t7}+\alpha_{t8}DF_{t8} \\ \qquad +\alpha_{t2}'DF_{t2}'+\alpha_{t3}'DF_{t3}'+\alpha_{t4}'DF_{t4}')E \end{array} \end{array}\right)$$

therefore

$$\begin{aligned} &2\mathrm{Tr}(\alpha_t DF_t+DF_t\alpha_t) \\ =8\{\ &\alpha_{t2}DF_{t2}+\alpha_{t3}DF_{t3}+\alpha_{t4}DF_{t4}+\alpha_{t2}'DF_{t2}'+\alpha_{t3}'DF_{t3}'+\alpha_{t4}'DF_{t4} \\ &-2(\alpha_{t5}DF_{t5}-\alpha_{t6}DF_{t6}-\alpha_{t7}DF_{t7}-\alpha_{t8}DF_{t8})\}. \end{aligned}$$

Let

$$A_i=\left(\begin{array}{cc|cc} A_{i2} & A_{i3}+iA_{i4} & A_{i5}+A_{i6} & A_{i7}+iA_{i8} \\ A_{i3}-iA_{i4} & -A_{i2} & A_{i7}-iA_{i8} & A_{i5}-A_{i6} \\ \hline -A_{i5}+A_{i6} & A_{i7}+iA_{i8} & A_{i2}' & A_{i3}'+iA_{i4}' \\ A_{i7}-iA_{i8} & -A_{i5}-A_{i6} & A_{i3}'-iA_{i4}' & -A_{i2}' \end{array}\right) \quad (i=t,x,y,z)$$

$$\bar{\bar{A}}_i=\left(\begin{array}{cc|cc} -\bar{A}_{i2}' & -\bar{A}_{i3}'+i\bar{A}_{i4}' & \bar{A}_{i5}+\bar{A}_{i6} & \bar{A}_{i7}-i\bar{A}_{i8} \\ -\bar{A}_{i3}'-i\bar{A}_{i4}' & \bar{A}_{i2}' & \bar{A}_{i7}+i\bar{A}_{i8} & \bar{A}_{i5}-\bar{A}_{i6} \\ \hline -\bar{A}_{i5}+\bar{A}_{i6} & \bar{A}_{i7}-i\bar{A}_{i8} & -\bar{A}_{i2} & -\bar{A}_{i3}+i\bar{A}_{i4} \\ \bar{A}_{i7}+i\bar{A}_{i8} & -\bar{A}_{i5}-\bar{A}_{i6} & -\bar{A}_{i3}-i\bar{A}_{i4} & \bar{A}_{i2} \end{array}\right)$$

$$F_i=\left(\begin{array}{cc|cc} F_{i2} & F_{i3}+iF_{i4} & F_{i5}+F_{i6} & F_{i7}+iF_{i8} \\ F_{i3}-iF_{i4} & -F_{i2} & F_{i7}-iF_{i8} & F_{i5}-F_{i6} \\ \hline -F_{i5}+F_{i6} & F_{i7}+iF_{i8} & F_{i2}' & F_{i3}'+iF_{i4}' \\ F_{i7}-iF_{i8} & -F_{i5}-F_{i6} & F_{i3}'-iF_{i4}' & -F_{i2}' \end{array}\right), \quad (i=t,x,y,z)$$

then

$$\begin{aligned} &\mathrm{Tr}([\bar{\bar{A}}_{ii},\alpha_t]\ F_i+F_i\ [\bar{\bar{A}}_i,\alpha_t])\ (i=t,x,y,z) \\ =&8(i\bar{A}_{i3}'\alpha_{t4}+i\bar{A}_{i4}'\alpha_{t3}+\bar{A}_{i5}\alpha_{t6}-\bar{A}_{i6}\alpha_{t5}-i\bar{A}_{i7}\alpha_{t8}-i\bar{A}_{i8}\alpha_{t7})\ F_{i2} \\ &+8(-i\bar{A}_{i2}'\alpha_{t4}-i\bar{A}_{i4}'\alpha_{t2}+\bar{A}_{i5}\alpha_{t7}+i\bar{A}_{i6}\alpha_{t8}-\bar{A}_{i7}\alpha_{t5}+i\bar{A}_{i8}\alpha_{t6})\ F_{i3} \\ &-8i(-\bar{A}_{i2}'\alpha_{t3}+\bar{A}_{i3}'\alpha_{t2}+i\bar{A}_{i5}\alpha_{t8}+\bar{A}_{i6}\alpha_{t7}-\bar{A}_{i7}\alpha_{t6}+i\bar{A}_{i8}\alpha_{t5})\ F_{i4} \\ &-8((-\bar{A}_{i2}'\alpha_{t6}-\bar{A}_{i3}'\alpha_{t7}+\bar{A}_{i4}'\alpha_{t8})-(\alpha_{t2}\bar{A}_{i6}+\alpha_{t3}\bar{A}_{i7}-\alpha_{t4}\bar{A}_{i8}) \\ &\qquad\qquad +(\bar{A}_{i6}\alpha_{t2}'+\bar{A}_{i7}\alpha_{t3}'-\bar{A}_{i8}\alpha_{t4}')-(-\alpha_{t6}\bar{A}_{i2}-\alpha_{t7}\bar{A}_{i3}+\alpha_{t8}\bar{A}_{i4}))\ F_{i5} \end{aligned}$$

$$
\begin{aligned}
&+8((-\bar{A}_{l2}'\alpha_{t5}+i\bar{A}_{l3}'\alpha_{t8}+i\bar{A}_{l4}'\alpha_{t7})-(\alpha_{l2}\bar{A}_{t5}+i\alpha_{l3}\bar{A}_{t8}+i\alpha_{l4}\bar{A}_{t7})\\
&\qquad+(\bar{A}_{l5}\alpha_{t2}'-i\bar{A}_{l7}\alpha_{t4}'-i\bar{A}_{l8}\alpha_{t3}')-(-\alpha_{t5}\bar{A}_{l2}-i\alpha_{t7}\bar{A}_{l4}-i\alpha_{t8}\bar{A}_{l3}))\ F_{l6}\\
&+8((-i\bar{A}_{l2}'\alpha_{t8}-\bar{A}_{l3}'\alpha_{t5}-i\bar{A}_{l4}'\alpha_{t6})-(-i\alpha_{t2}\bar{A}_{l8}+\alpha_{t3}\bar{A}_{l5}-i\alpha_{t4}\bar{A}_{l6})\\
&\qquad+(\bar{A}_{l5}\alpha_{t3}'+i\bar{A}_{l6}\alpha_{t4}'+i\bar{A}_{l8}\alpha_{t2}')-(-\alpha_{t5}\bar{A}_{l3}+i\alpha_{t6}\bar{A}_{l4}+i\alpha_{t8}\bar{A}_{l2}))\ F_{l7}\\
&-8i((-\bar{A}_{l2}'\alpha_{t7}+\bar{A}_{l3}'\alpha_{t6}+i\bar{A}_{l4}'\alpha_{t5})-(\alpha_{t2}\bar{A}_{l7}-\alpha_{t3}\bar{A}_{l6}+i\alpha_{t4}\bar{A}_{l5})\\
&\qquad+(i\bar{A}_{l5}\alpha_{t4}'+\bar{A}_{l6}\alpha_{t3}'-\bar{A}_{l7}\alpha_{t2}')-(i\alpha_{t5}\bar{A}_{l4}-\alpha_{t6}\bar{A}_{l3}+\alpha_{t7}\bar{A}_{l2}))\ F_{l8}\\
&+8(-\bar{A}_{l5}\alpha_{t6}+\bar{A}_{l6}\alpha_{t5}-i\bar{A}_{l7}\alpha_{t8}-i\bar{A}_{l8}\alpha_{t7}+i\bar{A}_{l3}\alpha_{t4}'+i\bar{A}_{l4}\alpha_{t3}')\ F_{l2}'\\
&+8(-\bar{A}_{l5}\alpha_{t7}+i\bar{A}_{l6}\alpha_{t8}+\bar{A}_{l7}\alpha_{t5}+i\bar{A}_{l8}\alpha_{t6}-i\bar{A}_{l2}\alpha_{t4}'-i\bar{A}_{l4}\alpha_{t2}')\ F_{l3}'\\
&-8i(-i\bar{A}_{l5}\alpha_{t8}+\bar{A}_{l6}\alpha_{t7}-\bar{A}_{l7}\alpha_{t6}-i\bar{A}_{l8}\alpha_{t5}-\bar{A}_{l2}\alpha_{t3}'+\bar{A}_{l3}\alpha_{t2}')\ F_{l4}'\\
=&\,\alpha_{t2}(8i(-\bar{A}_{l3}'F_{l4}-\bar{A}_{l4}'F_{l3})+8(\bar{A}_{l6}F_{l5}-\bar{A}_{l5}F_{l6}+i\bar{A}_{l8}F_{l7}+i\bar{A}_{l7}F_{l8}))\\
&+\alpha_{t3}(-8i(-\bar{A}_{l2}'F_{l4}-\bar{A}_{l4}'F_{l2})+8(\bar{A}_{l7}F_{l5}-i\bar{A}_{l8}F_{l6}-\bar{A}_{l5}F_{l7}-i\bar{A}_{l6}F_{l8}))\\
&+\alpha_{t4}(8i(-\bar{A}_{l2}'F_{l3}+\bar{A}_{l3}'F_{l2})+8(-\bar{A}_{l8}F_{l5}-i\bar{A}_{l7}F_{l6}+i\bar{A}_{l6}F_{l7}-\bar{A}_{l5}F_{l8}))\\
&+\alpha_{t5}(8(-\bar{A}_{l6}F_{l2}-\bar{A}_{l7}F_{l3}+\bar{A}_{l8}F_{l4})+8(-\bar{A}_{l2}'F_{l6}+\bar{A}_{l2}F_{l6}-\bar{A}_{l3}'F_{l7}+\bar{A}_{l3}F_{l7}+\bar{A}_{l4}'F_{l8}-\bar{A}_{l4}F_{l8})\\
&\qquad+8(\bar{A}_{l6}F_{l2}'+\bar{A}_{l7}F_{l3}'-\bar{A}_{l8}F_{l4}')\\
&+\alpha_{t6}(8(\bar{A}_{l5}F_{l2}+i\bar{A}_{l8}F_{l3}+i\bar{A}_{l7}F_{l4})+8(\bar{A}_{l2}'F_{l5}-\bar{A}_{l2}F_{l5}-i\bar{A}_{l4}'F_{l7}-i\bar{A}_{l4}F_{l7}-i\bar{A}_{l3}'F_{l8}-i\bar{A}_{l3}F_{l8})\\
&\qquad+8(-\bar{A}_{l5}F_{l2}'+i\bar{A}_{l8}F_{l3}'+i\bar{A}_{l7}F_{l4}')\\
&+\alpha_{t7}(8(-i\bar{A}_{l8}F_{l2}+\bar{A}_{l5}F_{l3}-i\bar{A}_{l6}F_{l4})+8(\bar{A}_{l3}'F_{l5}-\bar{A}_{l3}F_{l5}+i\bar{A}_{l4}'F_{l6}+i\bar{A}_{l4}F_{l6}+i\bar{A}_{l2}'F_{l8}+i\bar{A}_{l2}F_{l8})\\
&\qquad+8(-i\bar{A}_{l8}F_{l2}'-\bar{A}_{l5}F_{l3}'-i\bar{A}_{l6}F_{l4}')\\
&+\alpha_{t8}(8(-i\bar{A}_{l7}F_{l2}+i\bar{A}_{l6}F_{l3}+\bar{A}_{l5}F_{l4})+8(-\bar{A}_{l4}'F_{l5}+\bar{A}_{l4}F_{l5}+\bar{A}_{l3}'F_{l6}+i\bar{A}_{l3}F_{l6}-i\bar{A}_{l2}'F_{l7}-i\bar{A}_{l2}F_{l7})\\
&\qquad+8(-i\bar{A}_{l7}F_{l2}'+i\bar{A}_{l6}F_{l3}'-\bar{A}_{l5}F_{l4}')\\
&+\alpha_{t2}'(8(-\bar{A}_{l6}F_{l5}+\bar{A}_{l5}F_{l6}+i\bar{A}_{l8}F_{l7}+i\bar{A}_{l7}F_{l8})+8i(\bar{A}_{l4}F_{l3}'+\bar{A}_{l3}F_{l4}')\\
&+\alpha_{t3}'(8(-\bar{A}_{l7}F_{l5}-i\bar{A}_{l8}F_{l6}+\bar{A}_{l5}F_{l7}-i\bar{A}_{l6}F_{l8})-8i(\bar{A}_{l4}F_{l2}'+\bar{A}_{l2}F_{l4}')\\
&+\alpha_{t4}'(8(\bar{A}_{l8}F_{l5}-i\bar{A}_{l7}F_{l6}+i\bar{A}_{l6}F_{l7}+\bar{A}_{l5}F_{l8})-8i(\bar{A}_{l3}F_{l2}'-\bar{A}_{l2}F_{l3}').
\end{aligned}
$$

Let $\varepsilon_t=-1$ and $\varepsilon_l=1(i=x,y,z)$, then the follwing formulas hold , because the coefficients of α's is all zero , i, e. ,

$$\alpha_{t2}: DF_2+\sum_{l=t,x,y,z}2\varepsilon_l\ \{-i(\bar{A}_{l3}'F_{l4}+\bar{A}_{l4}'F_{l3})+(\bar{A}_{l6}F_{l5}-\bar{A}_{l5}F_{l6}+i\bar{A}_{l8}F_{l7}+i\bar{A}_{l7}F_{l8})\}$$

$$\alpha_{t3}: DF_3+\sum_{l=t,x,y,z}2\varepsilon_l\ \{i(\bar{A}_{l2}'F_{l4}+\bar{A}_{l4}'F_{l2})+(\bar{A}_{l7}F_{l5}-i\bar{A}_{l8}F_{26}-\bar{A}_{l5}F_{l7}-i\bar{A}_{l6}F_{l8})\}$$

$$\alpha_{t4}: DF_4+\sum_{l=t,x,y,z}2\varepsilon_l\ \{-i(\bar{A}_{l2}'F_{l3}-\bar{A}_{l3}'F_{l2})+(-\bar{A}_{l8}F_{l5}-i\bar{A}_{l7}F_{l6}+i\bar{A}_{l6}F_{l7}-\bar{A}_{l5}F_{l8})\}$$

$$
\begin{aligned}
\alpha_{t5}: -DF_5+\sum_{l=t,x,y,z}\varepsilon_l\ \{&(-\bar{A}_{l6}F_{l2}-\bar{A}_{l7}F_{l3}+\bar{A}_{l8}F_{l4})+(-\bar{A}_{l2}'F_{l6}+\bar{A}_{l2}F_{l6}-\bar{A}_{l3}'F_{l7}+\bar{A}_{l3}F_{l7}+\bar{A}_{l4}'F_{l8}-\bar{A}_{l4}F_{l8})\\
&+(\bar{A}_{l6}F_{l2}'+\bar{A}_{l7}F_{l3}'-\bar{A}_{l8}F_{l4}')\}
\end{aligned}
$$

$$
\begin{aligned}
\alpha_{t6}: DF_6+\sum_{l=t,x,y,z}\varepsilon_l\ \{&(\bar{A}_{l5}F_{l2}+i\bar{A}_{l8}F_{l3}+i\bar{A}_{l7}F_{l4})+(\bar{A}_{l2}'F_{l5}-\bar{A}_{l2}F_{l5}-i\bar{A}_{l4}'F_{l7}-i\bar{A}_{l4}F_{l7}-i\bar{A}_{l3}'F_{l8}-i\bar{A}_{l3}F_{l8})\\
&+(-\bar{A}_{l5}F_{l2}'+i\bar{A}_{l8}F_{l3}'+i\bar{A}_{l7}F_{l4}')\}
\end{aligned}
$$

$$
\begin{aligned}
\alpha_{t7}: DF_7+\sum_{l=t,x,y,z}\varepsilon_l\ \{&(-i\bar{A}_{l8}F_{l2}+\bar{A}_{l5}F_{l3}-i\bar{A}_{l6}F_{l4})+(\bar{A}_{l3}'F_{l5}-\bar{A}_{l3}F_{l5}+i\bar{A}_{l4}'F_{l6}+i\bar{A}_{l4}F_{l6}+\bar{A}_{l2}'F_{l8}+i\bar{A}_{l2}F_{l8})\\
&+(-i\bar{A}_{l8}F_{l2}'-\bar{A}_{l5}F_{l3}'-i\bar{A}_{l6}F_{l4}')\}
\end{aligned}
$$

$$
\begin{aligned}
\alpha_{t8}: DF_8+\sum_{l=t,x,y,z}\varepsilon_l\ \{&(-i\bar{A}_{l7}F_{l2}+i\bar{A}_{l6}F_{l3}+\bar{A}_{l5}F_{l4})+(-\bar{A}_{l4}'F_{l5}+\bar{A}_{l4}F_{l5}+i\bar{A}_{l3}'F_{l6}+i\bar{A}_{l3}F_{l6}-i\bar{A}_{l2}'F_{l7}-i\bar{A}_{l2}F_{l7})\\
&+(-i\bar{A}_{l7}F_{l2}'+i\bar{A}_{l6}F_{l3}'-\bar{A}_{l5}F_{l4}')\}
\end{aligned}
$$

$$\alpha_{t2}': DF_2'+\sum_{l=t,x,y,z}2\varepsilon_l\ \{(-\bar{A}_{l6}F_{l5}+\bar{A}_{l5}F_{l6}+i\bar{A}_{l8}F_{l7}+i\bar{A}_{l7}F_{l8})+i(\bar{A}_{l4}F_{l3}'+\bar{A}_{l3}F_{l4}')\}$$

$$\alpha_{t3}': DF_3'+\sum_{l=t,x,y,z}2\varepsilon_l\ \{(-\bar{A}_{l7}F_{l5}-i\bar{A}_{l8}F_{l6}+\bar{A}_{l5}F_{l7}-i\bar{A}_{l6}F_{l8})-i(\bar{A}_{l4}F_{l2}'+\bar{A}_{l2}F_{l4}')\}$$

α_{t4}' : $DF_4' + \sum_{l=t,x,y,z} 2\varepsilon_l \{(\bar{A}_{l8}F_{l5} - i\bar{A}_{l7}F_{l6} + i\bar{A}_{l6}F_{l7} + \bar{A}_{l5}F_{l8}) - i(\bar{A}_{l3}F_{l2}' - \bar{A}_{l2}F_{l3}')\}$

$$\begin{pmatrix} DF_t & 0 \\ 0 & DF_t \end{pmatrix} + \widetilde{\mathrm{Re}}\left(\begin{pmatrix} \bar{\bar{A}}_t - \bar{\bar{A}}_x & -\bar{\bar{A}}_y - i\bar{\bar{A}}_z \\ -\bar{\bar{A}}_y + i\bar{\bar{A}}_z & \bar{\bar{A}}_t + \bar{\bar{A}}_x \end{pmatrix} \cdot \begin{pmatrix} F_t + F_x & F_y + iF_z \\ F_y - iF_z & F_t - F_x \end{pmatrix} \right) = 0 .$$

The coefficient of α_x(i.e., $\alpha_t = \alpha_y = \alpha_z = 0$) is as follows:

$2\,\mathrm{Tr}(-\alpha_x DF_x - DF_x \alpha_x$

$+ [\bar{\bar{A}}_t, \alpha_x]\, F_t + F_t\, [\bar{\bar{A}}_t, \alpha_x] - [\bar{\bar{A}}_x, \alpha_x]\, F_x - F_x\, [\bar{\bar{A}}_x, \alpha_x] + i[\bar{\bar{A}}_y, \alpha_x]\, F_y + iF_y\, [\bar{\bar{A}}_y, \alpha_x]$

$- i[\bar{\bar{A}}_z, \alpha_x]\, F_z - iF_z\, [\bar{\bar{A}}_z, \alpha_x])$

therefore

$$\begin{pmatrix} DF_x & 0 \\ 0 & DF_x \end{pmatrix} + \widetilde{\mathrm{Im}}_x\left(\begin{pmatrix} \bar{\bar{A}}_t - \bar{\bar{A}}_x & -\bar{\bar{A}}_y - i\bar{\bar{A}}_z \\ -\bar{\bar{A}}_y + i\bar{\bar{A}}_z & \bar{\bar{A}}_t + \bar{\bar{A}}_x \end{pmatrix} \cdot \begin{pmatrix} F_t + F_x & F_y + iF_z \\ F_y - iF_z & F_t - F_x \end{pmatrix} \right) = 0 .$$

The coefficient of α_x(i.e., $\alpha_t = \alpha_x = \alpha_z = 0$) is as follows:

$2\,\mathrm{Tr}(-\alpha_y DF_y - DF_y \alpha_y$

$+ [\bar{\bar{A}}_t, \alpha_y]\, F_t + F_t\, [\bar{\bar{A}}_t, \alpha_y] - i[\bar{\bar{A}}_x, \alpha_y]\, F_x - iF_x\, [\bar{\bar{A}}_x, \alpha_y] - [\bar{\bar{A}}_y, \alpha_y]\, F_y - F_y\, [\bar{\bar{A}}_y, \alpha_y]$

$+ i[\bar{\bar{A}}_z, \alpha_y]\, F_z + iF_z\, [\bar{\bar{A}}_z, \alpha_y])$

therefore

$$\begin{pmatrix} 0 & DF_y \\ DF_y & 0 \end{pmatrix} + \widetilde{\mathrm{Im}}_y\left(\begin{pmatrix} \bar{\bar{A}}_t - \bar{\bar{A}}_x & -\bar{\bar{A}}_y - i\bar{\bar{A}}_z \\ -\bar{\bar{A}}_y + i\bar{\bar{A}}_z & \bar{\bar{A}}_t + \bar{\bar{A}}_x \end{pmatrix} \cdot \begin{pmatrix} F_t + F_x & F_y + iF_z \\ F_y - iF_z & F_t - F_x \end{pmatrix} \right) = 0 .$$

The coefficient of α_z(i.e., $\alpha_t = \alpha_x = \alpha_y = 0$) is as follows:

$2\,\mathrm{Tr}(-\alpha_z DF_z - DF_z \alpha_z$

$+ [\bar{\bar{A}}_t, \alpha_z]\, F_t + F_t\, [\bar{\bar{A}}_t, \alpha_z] + i[\bar{\bar{A}}_x, \alpha_z]\, F_x + iF_x\, [\bar{\bar{A}}_x, \alpha_z] - i[\bar{\bar{A}}_y, \alpha_z]\, F_y - iF_y\, [\bar{\bar{A}}_y, \alpha_z]$

$- [\bar{\bar{A}}_z, \alpha_z]\, F_z - F_z\, [\bar{\bar{A}}_z, \alpha_z])$

therefore

$$\begin{pmatrix} 0 & iDF_z \\ -iDF_z & 0 \end{pmatrix} + \widetilde{\mathrm{Im}}_z\left(\begin{pmatrix} \bar{\bar{A}}_t - \bar{\bar{A}}_x & -\bar{\bar{A}}_y - i\bar{\bar{A}}_z \\ -\bar{\bar{A}}_y + i\bar{\bar{A}}_z & \bar{\bar{A}}_t + \bar{\bar{A}}_x \end{pmatrix} \cdot \begin{pmatrix} F_t + F_x & F_y + iF_z \\ F_y - iF_z & F_t - F_x \end{pmatrix} \right) = 0 .$$

The above equation are in all

$$DF(A) + [\widetilde{\bar{\bar{A}}} \cdot F(A)] = 0 ,$$

this is the conditon of the Yang-Mills' connection.

§4 A example of gravitational field

Let (v^0, v^1, v^2, v^3) be the 4-dimensional velocity, i.e.,

$$(dc\tau)^2 = (dct)^2 - (dx)^2 - (dy)^2 - (dz)^2$$

and
$$v^0 = dct/dc\tau = c/(c^2-u^2)^{1/2},$$
$$v^1 = dx/dc\tau = x_t/(c^2-u^2)^{1/2}$$
$$v^2 = dy/dc\tau = y_t/(c^2-u^2)^{1/2}$$
$$v^3 = dz/dc\tau = z_t/(c^2-u^2)^{1/2}$$

where $u^2 = (x_t)^2 + (y_t)^2 + (z_t)^2$.

When a particle has a velocity $u = (x_t, y_t, z_t) = (x_t, 0, 0)$, the coordinate transformation in space-time is

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} v^0ct - v^1x \\ -v^1ct + v^0x \\ y \\ z \end{pmatrix} = \begin{pmatrix} v^0 & -v^1 & 0 & 0 \\ -v^1 & v^0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix},$$

and this is a Lorentz ($SO(1,3)$-) transformation (SPECIAL THEORY OF RELATIVITY).

Therefore we look upon $u = (x_t, y_t, z_t)$ as a parameter of the Lorentz transformation $L(u)$. and when the velocity change as the time going (ACCELERATION), we assign the different transformation $L(x_t, y_t, z_t)$ at each point on the locus (world line) of the particle.

We correspond this Lorentz group in space-time to the gauge subgroup in extended Hopf fiber bundle as follows[6]:

The transformation $SL(2, \mathbb{C})$ on $M_2(\mathbb{C})^2$ is

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} aX+bY \\ cX+dY \end{pmatrix}.$$

The gauge group $Sp(2, \mathbb{C})$ preserve the extended Hopf fiber bundle and acts on the base space such as when $|Y| \neq 0$

$$u = XY^{-1} \to u' = (aX+bY)(cX+dY)^{-1}$$
$$= (au+b)(cu+d)^{-1}.$$

Specially when the gauge subgroup $SL(2, \mathbb{C})' = \begin{pmatrix} a & 0 \\ 0 & {}^t\bar{a}^{-1} \end{pmatrix}$, $a \in SL(2, \mathbb{C})$

acts as the Lorentz transformation on the $R^{1,3}$ (Minkowski space), i.e.,

$$u \to u' = au({}^t\bar{a}^{-1})^{-1}$$
$$= au{}^t\bar{a}$$

where we look upon $u = \begin{pmatrix} ct+x & y+iz \\ y-iz & ct-x \end{pmatrix}$ as the vector $\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$.

By this 2-1 correspondence σ: $SL(2, \mathbb{C})' \to SO(1,3)$, we denote the above gauge transformation as $\pm L'(x_t, y_t, z_t)$.

and we assign the gauge transformation $L'(x_t, y_t, z_t)$ at each point on the world line of the particle.

(Does the gauge transformation $SL(2, \mathbb{C})'$ generate the gravitation ?)

We assign the gauge transformation $L'(x_t, t_t, z_t)$ on each point (ct, x, y, z) of space-time.

Then the potential is

$$
\begin{aligned}
A &= (L')^{-1} DL' \\
&= (L')^{-1}(d_{ct}(L')dct + d_x(L')dx + d_y(L')dy + d_z(L')dz) \\
&= d_{ct}(\log L')dct + d_x(\log L')dx + d_y(\log L')dy + d_z(\log L')dz
\end{aligned}
$$

therefore the curvature is

$$
\begin{aligned}
F &= dA + \tfrac{1}{2}[A \wedge A] \qquad (\because\ sl(2, \mathbb{C})' \text{ is a real under the conjugate } "=") \\
&= d((L')^{-1}(d_{ct}(L'))\wedge dct + d((L')^{-1}(dx(L'))\wedge dx + d((L')^{-1}(d_y(L'))\wedge dy + d((L')^{-1}(d_z(L'))\wedge dz \\
&\quad + ((L')^{-1}(d_{ct}(L')dct + dx(L')dx + d_y(L')dy + d_z(L')dz))\wedge \\
&\qquad ((L')^{-1}(d_{ct}(L')dct + d_x(L')dx + d_y(L')dy + d_z(L')dz) \\
&= 0
\end{aligned}
$$

and the new factor of the curvature is

$$
\begin{aligned}
E_t &= d_{ct}((L')^{-1}d_{ct}(L')) + d_x((L')^{-1}d_x(L')) + d_y((L')^{-1}d_y(L')) + d_z((L')^{-1}d_z(L')) \\
&= d^2{}_{ct}(\log L') + d^2{}_x(\log L') + d^2{}_y(\log L') + d^2{}_z(\log L')
\end{aligned}
$$

therefore when log L' is harmonic then the curvature matrix is zero.

This means that the gravitational field is free, when we have any choice of the gauge group such that log L' is harmonic.

If we mention the other word , we must start from the potential and infinitesimal gauge transformation in order to generate the "real" gravitational field.

We base upon above argument and introduce the potential of a particle with gravitational mass m_G as follows:

$$
A = \frac{m_G}{r}\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} dx + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} dy + \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} dz\right) \in sl(2, \mathbb{C})'
$$

and these coefficients of dx, dy, dz correspond to the infinitesimal Lorentz transformation as

$$
\frac{2m_G}{r}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \frac{2m_G}{r}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \frac{2m_G}{r}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}
$$

because the gravity of a particle carry the x-directional acceleration at the point which separate from the particle to the x-direction and the same for the y, z-direction.

This potential does not satisfy the Yang-Mills' equation (in Corollary 4).

Therefore we take the gauge subgroup $SL(2, \mathbb{C})'' = \begin{pmatrix} a & 0 \\ 0 & E \end{pmatrix}$, $a \in SL(2, \mathbb{C})$ instead of $SL(2, \mathbb{C})'$,

and it acts on the base space as a complex Lorentz transformation $SO(1,3,\mathbb{C})$ on the $\mathbb{C}^{1,3}$ space , i.e.,

$$
u \quad \to \quad u' = au
$$

Under the above consideration , we introduce the potential of a particle as follows:

$$A = 1/c\cdot\phi\ dct - A_x\ dx - A_y\ dy - A_z\ dz$$

$$= \frac{m_G}{r}\left\{ \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) dx + \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) dy + \left(\begin{array}{cc|cc} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) dz \right\} \in sl(2, \mathbb{C})\text{”}$$

and these coefficients of dx,dy,dz correspond to the infinitesimal transformations on $\mathbb{C}^{1,3}$ as

$$\frac{m_G}{r}\left(\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{array}\right), \frac{m_G}{r}\left(\begin{array}{c|ccc} 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & -i \\ 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{array}\right) \text{ and } \frac{m_G}{r}\left(\begin{array}{c|ccc} 0 & 0 & 0 & 1 \\ \hline 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right).$$

Then its curvature F is

$$E_x = \partial A_x/\partial t - 1/c\cdot\partial\phi/\partial x + \tfrac{1}{2}c\cdot([\bar{\bar{\phi}}, A_x] - [\bar{\bar{A}}_x, \phi]) = 0$$

$$E_y = \partial A_y/\partial t - 1/c\cdot\partial\phi/\partial y + \tfrac{1}{2}c\cdot([\bar{\bar{\phi}}, A_y] - [\bar{\bar{A}}_y, \phi]) = 0$$

$$E_z = \partial A_z/\partial t - 1/c\cdot\partial\phi/\partial z + \tfrac{1}{2}c\cdot([\bar{\bar{\phi}}, A_z] - [\bar{\bar{A}}_z, \phi]) = 0$$

$$B_x = -\partial A_z/\partial y + \partial A_y/\partial z + \tfrac{1}{2}([\bar{\bar{A}}_y, A_z] - [\bar{\bar{A}}_z, A_y])$$

$$= -\frac{m_G}{r^3}\left(\begin{array}{cc|cc} 0 & z-iy & 0 & 0 \\ z+iy & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$B_y = -\partial A_x/\partial z + \partial A_z/\partial x + \tfrac{1}{2}([\bar{\bar{A}}_z, A_x] - [\bar{\bar{A}}_x, A_z])$$

$$= -\frac{m_G}{r^3}\left(\begin{array}{cc|cc} -z & ix & 0 & 0 \\ -ix & z & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$B_z = -\partial A_y/\partial x + \partial A_x/\partial y + \tfrac{1}{2}([\bar{\bar{A}}_x, A_y] - [\bar{\bar{A}}_y, A_x])$$

$$= -\frac{m_G}{r^3}\left(\begin{array}{cc|cc} y & -x & 0 & 0 \\ -x & -y & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

and a new factor is

$$E_t = \partial A_x/\partial x + \partial A_y/\partial y + \partial A_z/\partial z + 1/c\cdot\partial\phi/\partial ct$$
$$+ \tfrac{1}{2}([\bar{\bar{A}}_t, A_t] - [\bar{\bar{A}}_x, A_x] - [\bar{\bar{A}}_y, A_y] - [\bar{\bar{A}}_z, A_z])$$

$$= \frac{m_G}{r^3}\left(\begin{array}{cc|cc} x & y+iz & 0 & 0 \\ y-iz & -x & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The above curvature and a new factor satisfy the Yang-Mills' equation ,i.e.,

$$\rho = \partial E_t/\partial t + (\partial E_x/\partial x + \partial E_y/\partial y + \partial E_z/\partial z)$$
$$+ [\bar{\bar{\phi}}, E_t] - [\bar{\bar{A}}_x, E_x] - [\bar{\bar{A}}_y, E_y] - [\bar{\bar{A}}_z, E_z]$$
$$= 0$$

$$j_x = (\partial E_x/\partial t + \partial E_t/\partial x) - (\partial B_y/\partial z - \partial B_z/\partial y)$$

$$+\ [\bar{\bar{\phi}}, E_x] - [\bar{\bar{A}}_x, E_t] - [\bar{\bar{A}}_y, B_z] + [\bar{\bar{A}}_z, B_y]$$

$$= \frac{3m_G}{r^5}\left(\begin{array}{cc|cc} x^2+y^2+z^2 & xy+ixz\text{-}iz\text{-}yx & 0 & 0 \\ xy\text{-}ixz+izx\text{-}yx & \text{-}x^2\text{-}x^2\text{-}z^2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) + \frac{m_G}{r^3}\left(\begin{array}{cc|cc} 1+1+1 & 0 & 0 & 0 \\ 0 & \text{-}1\text{-}1\text{-}1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$=0$$

$$j_y = (\partial E_y/\partial t + \partial E_t/\partial y) - (\partial B_z/\partial x - \partial B_x/\partial z)$$

$$+\ [\bar{\bar{\phi}}, E_y] + [\bar{\bar{A}}_x, B_z] - [\bar{\bar{A}}_y, E_t] - [\bar{\bar{A}}_z, B_x]$$

$$= -\frac{3m_G}{r^5}\left(\begin{array}{cc|cc} yx\text{-}xy & x^2iyz+y^2+z^2\text{-}izy & 0 & 0 \\ x^2\text{-}iyz+y^2+y^2+z^2+izy & \text{-}yx+xy & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) + \frac{m_G}{r^3}\left(\begin{array}{cc|cc} 0 & 1+1+1 & 0 & 0 \\ 1+1+1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$=0$$

$$j_z = (\partial E_z/\partial t + \partial E_t/\partial z) - (\partial B_x/\partial y - \partial B_y/\partial x)$$

$$+\ [\bar{\bar{\phi}}, E_z] - [\bar{\bar{A}}_x, B_y] + [\bar{\bar{A}}_y, B_x] - [\bar{\bar{A}}_z, E_t]$$

$$= -\frac{3m_G}{r^5}\left(\begin{array}{cc|cc} zx\text{-}xz & zy+iz^2\text{-}yz+iy^2+ix^2 & 0 & 0 \\ zy\text{-}iz^2\text{-}yz\text{-}iy^2\text{-}ix^2 & \text{-}zx+xz & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) + \frac{m_G}{r^3}\left(\begin{array}{cc|cc} 0 & i+i+i & 0 & 0 \\ \text{-}i\text{-}i\text{-}i & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$=0$$

$$\rho' = \partial B_x/\partial x + \partial B_y/\partial y + \partial B_z/\partial z + [\bar{\bar{\phi}}, B_t] - [\bar{\bar{A}}_x, B_x] - [\bar{\bar{A}}_y, B_y] - [\bar{\bar{A}}_z, B_z] = 0$$

$$j'_x = \partial B_x/\partial t + (\partial E_y/\partial z - \partial E_z/\partial y) + [\bar{\bar{\phi}}, B_x] - [\bar{\bar{A}}_x, B_t] + [\bar{\bar{A}}_y, E_z] - [\bar{\bar{A}}_z, E_y] = 0$$

$$j'_y = \partial B_y/\partial t + (\partial E_z/\partial x - \partial E_x/\partial z) + [\bar{\bar{\phi}}, B_y] - [\bar{\bar{A}}_x, E_z] - [\bar{\bar{A}}_y, B_t] + [\bar{\bar{A}}_z, E_x] = 0$$

$$j'_z = \partial B_z/\partial t + (\partial E_x/\partial y - \partial E_y/\partial x) + [\bar{\bar{\phi}}, B_z] + [\bar{\bar{A}}_x, E_y] - [\bar{\bar{A}}_y, E_x] - [\bar{\bar{A}}_z, B_t] = 0.$$

Let (x^0, x^1, x^2, x^3) and g_{ij} be a generalized coordinates and its metric on the Minkowski space. and $\Gamma^i{}_{jk}$ be a connection which is torsion free and compatible with the metric g_{ij}, then the covariant derivative of the vector field (u^0, u^1, u^2, u^3) is

$$\nabla_j u^i = \partial u^i/\partial x^j + \Gamma^i{}_{jk} u^k$$

moreover when the vector (u^i) is coincide with $(dx^i/\ dc\tau)$ which is the velocity of a particle on the world line.

Then

$$\nabla\ u^i = \partial u^i/\partial x^j \cdot dx^j + \Gamma^i{}_{jk} u^k \cdot dx^j\ ,\ \ \nabla\ u^i/dc\tau = 0$$

therefore

$$du^i/dc\tau\ =\ -\Gamma^i{}_{jk} u^j u^k.$$

This is a equation of motion and the right hand side means 4-dimensional force.

We rewrite the above equation of motion by using of the normal vector (u'^i) instead of (u^i), then

$$du'^i/dc\tau = -\Gamma'^i{}_{jk} u'^j u'^k,\ \ (\Gamma'^i{}_{jk})\ =\ (\Gamma'^i{}_k)_j\ \ \in so(1,3).$$

We notice that the electromagnetic field (curvature) generate a force and identify the curvature $G\sigma(E_t)$, $\pm G\sigma(E_x + iB_x)$, $\pm G\sigma(E_y + iB_y)$, $\pm G\sigma(E_z + iB_z)$ as the connection $(\Gamma'^i{}_k)_0$, $(\Gamma'^i{}_k)_1$, $(\Gamma'^i{}_k)_2$, $(\Gamma'^i{}_k)_3$ and the vector (v^i) as (u'^i).

For example, let

$$d\begin{pmatrix} v^0 \\ v^1 \\ v^2 \\ v^3 \end{pmatrix} = \text{-}G\cdot m_G(\sigma(E_t)dct\text{-}\sigma(E_x+iB_x)dx\text{-}\ \sigma(E_y+iB_y)dy\text{-}\ \sigma(E_z+iB_z)dz)\begin{pmatrix} v^0 \\ v^1 \\ v^2 \\ v^3 \end{pmatrix}$$

where G is a gravitational constant

,i.e.,

$$\frac{d}{dt}\begin{pmatrix} v^1 \\ v^2 \\ v^3 \\ v^4 \end{pmatrix} = \frac{\text{-}Gm_G}{r^3}\left\{\begin{pmatrix} 0 & x & y & z \\ x & 0 & iz & \text{-}iz \\ y & \text{-}iz & 0 & ix \\ z & iy & \text{-}ix & 0 \end{pmatrix} c + \begin{pmatrix} 0 & 0 & iz & \text{-}iy \\ 0 & 0 & y & z \\ iz & \text{-}y & 0 & 0 \\ \text{-}iy & \text{-}z & 0 & 0 \end{pmatrix}\cdot\frac{dx}{dt} + \begin{pmatrix} 0 & \text{-}iz & 0 & ix \\ \text{-}iz & 0 & \text{-}x & 0 \\ 0 & x & 0 & z \\ ix & 0 & \text{-}z & 0 \end{pmatrix}\cdot\frac{dy}{dt}\right.$$

$$\left. + \begin{pmatrix} 0 & iy & \text{-}ix & 0 \\ iy & 0 & 0 & \text{-}x \\ \text{-}ix & 0 & 0 & \text{-}y \\ 0 & x & y & 0 \end{pmatrix}\cdot\frac{dz}{dt}\right\}\cdot\begin{pmatrix} v^1 \\ v^2 \\ v^3 \\ v^4 \end{pmatrix}.$$

Then

(t-component)

$$dv^1/dt = -Gm_G\cdot r^{-3}\ \{(xv^2+yv^3+zv^4)c+(izv^3-iyv^4)x_t+(-izv^2+ixv^4)y_t+(iyv^2-ixv^3)z_t\}$$

$$d/dt\ \{c^2/(c^2-u^2)^{1/2}\} = -Gm_Gc^2\cdot r^{-3}(xx_t+yy_t+zz_t)/(c^2-u^2)^{1/2}$$
$$-Gm_Gc\cdot r^{-3}\ \{(izy_t-iyz_t)x_t+(-izx_t+ixz_t)y_t+(iyx_t-ixy_t)z_t\}/(c^2-u^2)^{1/2}$$

$$\frac{1}{2}\cdot c^2(u^2)_t/(c^2-u^2)^{3/2}\ -Gm_Gc^2\cdot r^{-2}r_t/(c^2-u^2)^{1/2}$$
$$+iGm_Gc\cdot r^{-3}\ \{(yz_t-zy_t)x_t+(zx_t-xz_t)y_t+(xy_t-yz_t)z_t\}/(c^2-u^2)^{1/2}$$

$$(mu^2)_t=2Gmm_G(c^2-u^2)\cdot(1/r)_t$$

$$(mu^2)_t=2Gmm_Gc^2\cdot(1/r)_t\quad(\text{when } u \ll 0),$$

therefore

$$\frac{1}{2}\,mu^2-Gmm_Gc^2/r = \text{constant}$$

where m is a mass of another particle.

(x-component)

$$dv^2/dt = -Gm_G\cdot r^{-3}\ \{(xv^1+izv^3-iyv^4)c+(yv^3+zv^4)x_t+(-izv^1-xv^3)y_t+(iyv^1-xv^4)z_t\}$$

$$d/dt\ \{cx_t/(c^2-u^2)^{1/2}\} = -Gm_G\cdot r^{-3}\ \{xc^2+(yy_t+zz_t)x_t-x(y_t)^2-x(z_t)^2\}/(c^2-u^2)^{1/2}$$
$$-iGm_Gc^2\cdot r^{-3}\ \{(zy_t-yz_t)-zy_t+yz_t\}/(c^2-u^2)^{1/2}$$

$$cx_{tt}/(c^2-u^2)^{1/2}+cx_tu_t(c^2-u^2)^{3/2} = -Gm_Gc\cdot r^{-3}\ \{xc^2+(yy_t+zz_t)x_t-x(u^2-(x_t)^2)\}/(c^2-u^2)^{1/2}$$

$$(c^2-u^2)x_{tt}+x_tu_t= -Gm_G(c^2-u^2)\cdot r^{-3}\ \{xc^2+(xx_t+yy_t+zz_t)x_t-xu^2\}$$
$$= -Gm_G(c^2\text{-}u^2)^2\cdot xr^{-3}\ -\ Gm_G(c^2-u^2)x_tr_tr^{-2}.$$

(y-component)

$$dv^3/dt = -Gm_G\cdot r^{-3}\ \{(yv^1-izv^2+ixv^4)c+(izv^1-yv^2)x_t+(xv^2+zv^4)y_t+(\text{-}ixv^1-yv^4)z_t\}$$

$$d/dt\ \{cy_t/(c^2-u^2)^{1/2}\} = -Gm_Gc\cdot r^{-3}\ \{yc^2+(zz_t+xx_t)y_t-y(x_t)^2-y(z_t)^2\}/(c^2-u^2)^{1/2}$$
$$-iGm_Gc^2\cdot r^{-3}\ \{(-zx_t+xz_t)+zx_t-xz_t\}/(c^2-u^2)^{1/2}$$

$$cy_{tt}/(c^2-u^2)^{1/2}+cy_tu_t/(c^2-u^2)^{3/2} = -Gm_Gc\cdot r^{-3}\ \{yc^2+(zz_t+xx_t)y_t-y(u^2-(y_t)^2)\}/(c^2-u^2)^{1/2}$$

$$(c^2-u^2)y_{tt}+y_tu_t= -Gm_G(c^2-u^2)\cdot r^{-3}\ \{yc^2+(xx_t+yy_t+zz_t)y_t-yu^2\}$$
$$= -Gm_G(c^2-u^2)^2\cdot yr^{-3}\ -\ Gm_G(c^2-u^2)y_tr_tr^{-2}.$$

(z-component)

$$dv^4/dt = -Gm_G\cdot r^{-3}\ \{(zv^1+iyv^2-ixv^3)c+(-iyv^1-zv^2)x_t+(ixv^1-zv^3)y_t+(xv^2+yv^3)z_t\}$$

$$d/dt\ \{cz_t/(c^2-u^2)^{1/2}\} = -Gm_Gc\cdot r^{-3}\ \{zc^2+(xx_t+yy_t)z_t-z(x_t)^2-z(y_t)^2\}/(c^2-u^2)^{1/2}$$
$$-iGm_Gc^2\cdot r^{-3}\ \{(yx_t-xy_t)-yx_t+xy_t\}/(c^2-u^2)^{1/2}$$

$$cz_{tt}/(c^2-u^2)^{1/2}+cz_tu_t/(c^2-u^2)^{3/2} = -Gm_Gc\cdot r^{-3}\ \{zc^2+(xx_t+yy_t)z_t-z(u^2-(z_t)^2)\}/(c^2-u^2)^{1/2}$$

$$(c^2-u^2)z_{tt}+z_tu_t = -Gm_G(c^2-u^2)\cdot r^{-3}\ \{zc^2+(xx_t+yy_t+zz_t)z_t-zu^2\}$$

$$= -Gm_G(c^2-u^2)^2\cdot zr^{-3} - Gm_G(c^2-u^2)z_tr_t r^{-2}.$$

The x, y, z— component formulas are in all

$$m(c^2-u^2)x_{tt} + mux_tu_t = -Gmm_G(c^2-u^2)^2\cdot xr^{-3} + Gmm_G(c^2-u^2)x_tr_tr^{-2}$$

$$m(c^2-u^2)y_{tt} + muy_tu_t = -Gmm_G(c^2-u^2)^2\cdot yr^{-3} + Gmm_G(c^2-u^2)y_tr_tr^{-2}$$

$$m(c^2-u^2)z_{tt} + muz_tu_t = -Gmm_G(c^2-u^2)^2\cdot zr^{-3} + Gmm_G(c^2-u^2)z_tr_tr^{-2},$$

therefore

$$m(c^2-u^2)\alpha+muUu_t = -Gmm_G(c^2-u^2)^2Rr^{-3} - Gmm_G(c^2-u^2)Ur_tr^{-2}$$

where α is an acceleration , U is a velocity , R is a position vector and each a, u, r is a length of α, U, R respectively.

This means that

$$m(c^2-u^2)\alpha = -Gmm_G(c^2-u^2)^2\cdot Rr^{-3}\ \therefore\ \alpha = -Gm_G(c^2-u^2)\cdot Rr^{-3}$$

$$muu_t = -Gmm_G(c^2-u^2)\cdot r_tr^{-2} \qquad \therefore\ (u^2)_t = 2Gm_G(c^2-u^2)\cdot(1/r)_t.$$

hold by comparison with the coefficients of U and the equation of "t-component".

Eventually, we obtain the Newton's equation of motion and the conservation of energy under the universal gravitation when the speed of another particle is very slow, i.e.,

when $u \ll 0$, then

$$m\alpha = -Gmm_Gc^2\cdot Rr^{-3} \quad \text{and} \quad \frac{1}{2}mu^2-G = \text{constant}.$$

We abbreviate the imaginary component in the first equation by the process of the above calculation. therefore we obtain the approximate form of Newton's equation as follows :

$$d\begin{pmatrix} v^1 \\ v^2 \\ v^3 \\ v^4 \end{pmatrix}\cdot\begin{pmatrix} v^1 \\ v^2 \\ v^3 \\ v^4 \end{pmatrix}^{-1} = \frac{-Gm_G}{r^3}\left(\left(\begin{array}{c|ccc} 0 & x & y & z \\ \hline x & 0 & 0 & 0 \\ y & 0 & 0 & 0 \\ z & 0 & 0 & 0 \end{array}\right)dct + \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & y & z \\ 0 & -y & 0 & 0 \\ 0 & -z & 0 & 0 \end{array}\right)dx + \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -x & 0 \\ 0 & x & 0 & z \\ 0 & 0 & -z & 0 \end{array}\right)dy\right.$$

$$\left. + \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -x \\ 0 & 0 & 0 & -y \\ 0 & x & y & 0 \end{array}\right)dz\right).$$

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