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An extension of Maxwell's equations and the deduction from a Yang-Mills functional*

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Abstract

We introduce the fourth component of the electromagnetic field and make the correspondence between the Maxwell's equations and the wave equation without additional Lorentz condition for scalar and vector potentials (§ 2). We rewrite the equations by the use of matrices (§ 3). Then, we can complexify the potential and revive the symmetry between the electric and the magnetic fields (§ 5). Moreover we show that the extended Maxwell's equations can be derived from a modified Yang-Mills functional (§ 4).

At last, we study the transformations of the matrices for the special theory of relativity (§ 6).

§ 1 Introduction

As is well known the conventional Maxwell's equations are as follows:
We take the light velocity c=1 for simplicity in the § 1 ∼§ 5.

$\mathrm{rot}\ E+\partial B/\partial t=0$	(Farday's law of induction)	(1. 1)
$\mathrm{div}\ B=0$	(No magnetic charges)	(1. 2)
$\mathrm{div}\ E=\rho$	(Gauss' law)	(1. 3)
$\mathrm{rot}\ B-\partial E/\partial t=J$	(Ampère's law)	(1. 4)

From eq. (1.2), exists the vector potencial A which satisfies

$$B=\mathrm{rot}\ A. \tag{1.5}$$

And from this eq. (1.5) and eq. (1.1),

$$\mathrm{rot}\ (E+\partial A/\partial t)=0.$$

Therefore exists the scalar potencial ϕ which satisfies

$$E=-\mathrm{grad}\ \phi-\partial A/\partial t. \tag{1.6}$$

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This electromagnetic field is invariant by the following gauge transformation

$\mathsf{A}' = \mathsf{A} - \mathrm{grad}\ \chi,$

$\phi' = \phi + \partial\chi/\partial t,$

(where χ is an arbitrary continuously differentiable function) .

We substitute the eqs. (1.5) and (1.6) into the eqs. (1.3) and (1.4), then we get the following formula

$$\partial^2\phi/\partial t^2 - \nabla^2\phi - \partial/\partial t\ (\underset{\sim\sim\sim\sim\sim\sim\sim\sim}{\mathrm{div}\ \mathsf{A} + \partial\phi/\partial t}) = \rho,$$

$$\partial^2\mathsf{A}/\partial t^2 - \nabla^2\mathsf{A} + \mathrm{grad}\ (\underset{\sim\sim\sim\sim\sim\sim\sim\sim}{\mathrm{div}\ \mathsf{A} + \partial\phi/\partial t}) = \mathsf{J}\,.$$

Moreover, in the above equations we add the following Lorentz condition

$$\mathrm{div}\ \mathsf{A} + \partial\phi/\partial t = 0\,,$$

then we get the following wave equations

$$\partial^2\phi/\partial t^2 - \nabla^2\phi = \rho, \qquad (1\,.\ 7)$$

$$\partial^2\mathsf{A}/\partial t^2 - \nabla^2\mathsf{A} = \mathsf{J}\,. \qquad (1\,.\ 8)$$

§2 The modification of Maxwell's equations

Here we say that we can remove the Lorentz condition by introducing the fourth component E_t of the electric field for the electromagnetic theory and derive the wave equations (1.7,8) only from the resultant Maxwell's equations even if the Lorentz condition is not satisfied.

Let the fourth component E_t be

$$\mathrm{E}_t = \mathrm{div}\ \mathsf{A} + \partial\phi/\partial t, \qquad (1\,.\ 6)_t$$

then we get the following modified Maxwell's equations

Theorem 1. (the modified Maxwell's equations)

$$\mathrm{rot}\ \mathsf{E} + \partial\mathsf{B}/\partial t = 0 \qquad (1\,.\ 1)$$

$$\mathrm{div}\ \mathsf{B} = 0 \qquad (1\,.\ 2)$$

$$\mathrm{div}\ \mathsf{E} + \underline{\partial\mathrm{E}_t/\partial t} = \rho \qquad (1\,.\ 3)'$$

$$\mathrm{rot}\ \mathsf{B} - \partial\mathsf{E}/\partial t - \underline{\mathrm{grad}\ \mathrm{E}_t} = \mathsf{J} \qquad (1\,.\ 4)'$$

(where the underlined parts are ones which are obtained from E_t)

From eqs. (1.1) and (1.2) , exist the scalar and vector potential ϕ, A, and

$$\mathsf{B} = \mathrm{rot}\ \mathsf{A}, \tag{1.5}$$
$$\mathsf{E} = -\mathrm{grad}\phi - \partial \mathsf{A}/\partial t, \tag{1.6}$$

hold.

This extended electromagnetic field is invariant by the following gauge transformation

$$\mathsf{A}' = \mathsf{A} - \mathrm{grad}\ \chi,$$
$$\phi' = \phi + \partial\chi/\partial t,$$

(where χ is an arbitrary harmonic function) .

We substitute eqs. (1.5) , (1.6) and $(1.6)_t$ into eqs. (1.3)' and (1.4)', then we get the following wave equations

$$\partial^2\phi/\partial t^2 - \nabla^2\phi = \rho, \tag{1.7}$$
$$\partial^2 \mathsf{A}/\partial t^2 - \nabla^2 \mathsf{A} = \mathsf{J} \tag{1.8}$$

Moreover, we assume that the charge and current are null, then we get the wave equations of the electromagnetic field as follows :

From eqs. (1,1) and (1,4)', we get

$$\begin{aligned} \partial^2 \mathsf{B}/\partial t^2 &= -\mathrm{rot}\ \partial \mathsf{E}/\partial t \\ &= -\mathrm{rot}(\mathrm{rot}\ \mathsf{B} - \mathrm{grad}\ E_t) \\ &= \nabla^2 \mathsf{B} \end{aligned} \tag{1.9}$$

and from eqs. (1,1) , (1.3)'and (1.4)', we get

$$\begin{aligned} \partial^2 \mathsf{E}/\partial t^2 &= \mathrm{rot}\ \partial \mathsf{B}/\partial t - \mathrm{grad}\ \partial E_t/\partial t \\ &= -\mathrm{rot}\ \mathrm{rot}\ \mathsf{E} - \mathrm{grad}\ \partial E_t/\partial t \\ &= \nabla^2 \mathsf{E} \end{aligned} \tag{1.10}$$

and from eqs. (1,1) and (1,3)', we get

$$\begin{aligned} \partial^2 E_t/\partial t^2 &= -\mathrm{div}\ \partial \mathsf{E}/\partial t \\ &= -\mathrm{div}(\mathrm{rot}\ \mathsf{B} - \mathrm{grad}\ E_t) \\ &= \nabla^2 E_t \end{aligned} \tag{1.10$_t$}$$

§ 3 The matrix representation of the modified Maxwell's equations.

We rewrite the modified Maxwell's equations by the matrices as follows :

Let the potential matrix be $\tilde{A}$, the field matrix F and the current matrix $\tilde{J}$.

,i.e.,

$$\tilde{A} = \begin{pmatrix} \phi - A_x & -A_y - iA_z \\ -A_y + iA_z & \phi + A_x \end{pmatrix}, \quad F = \begin{pmatrix} E_t + iB_t + E_x + iB_x & E_y + iB_y + iE_z - B_z \\ E_y + iB_y - iE_z + B_z & E_t + iB_t - E_x - iB_x \end{pmatrix},$$

$$\tilde{J} = \begin{pmatrix} \rho - j_x & -j_y - ij_z \\ -j_y + ij_z & \rho + j_x \end{pmatrix},$$

(where A_x, A_y, A_z and j_x, j_y, j_z are x, y, z-component of A and J respectively and $B_t = 0$) .

and let the <u>differential operator matrices</u> be $D, \tilde{D}$.
,i.e.,

$$D = \begin{pmatrix} \partial/\partial t + \partial/\partial x & \partial/\partial y + i\partial/\partial z \\ \partial/\partial y - i\partial/\partial z & \partial/\partial t - \partial/\partial x \end{pmatrix}, \quad \tilde{D} = \begin{pmatrix} \partial/\partial t - \partial/\partial x & -\partial/\partial y - i\partial/\partial z \\ -\partial/\partial y + i\partial/\partial z & \partial/\partial t + \partial/\partial x \end{pmatrix},$$

then we get the following equations.

Proposition 2. (the modified Maxwell's equation in matrix form)

$\tilde{J} = D \cdot F$ (The modified Maxwell's equation) (1.1), (1.2), (1.3)' and (1.4)'

is the extended Maxwell's equations, and exist a potential $\tilde{A}$, and

$F = \tilde{D} \cdot \tilde{A}$ (The existence of a potential) (1.5), (1.6) and $(1.6)_t$

hold.

Moreover the wave equation of $\tilde{A}$ is

$\tilde{J} = D \cdot \tilde{D} \cdot \tilde{A}$ (The wave equation of $\tilde{A}$) (1.7) and (1.8)

and the wave equation of F (when $\tilde{J} = 0$) is

$0 = \tilde{D} \cdot D \cdot F$ (The wave equation of F) (1.9), (1.10) and $(1.10)_t$

Proof.

From the equation $\tilde{J} = D \cdot F$, we may rewrite as

$$\begin{pmatrix} \rho - j_x & -j_y - ij_z \\ -j_y + ij_z & \rho + j_x \end{pmatrix} = \begin{pmatrix} \partial/\partial t + \partial/\partial x & \partial/\partial y + i\partial/\partial z \\ \partial/\partial y - i\partial/\partial z & \partial/\partial t - \partial/\partial x \end{pmatrix} \begin{pmatrix} E_t + E_x + iB_x & E_y + iB_y + iE_z - B_z \\ E_y + iB_y - iE_z + B_z & E_t - E_x - iB_x \end{pmatrix}$$

,i.e.,

$$\rho = \partial E_t/\partial t + E_x/\partial x + \partial E_y/\partial y + \partial E_z/\partial z \quad \Rightarrow \quad \mathrm{div}\mathbf{E} + \partial E_t/\partial t = \rho \qquad (1.3)'$$

$$\left.\begin{aligned} j_x &= (\partial B_y/\partial z - B_z/\partial y) - \partial E_x/\partial t - \partial E_t/\partial x \\ j_y &= (\partial B_z/\partial x - B_x/\partial z) - \partial E_y/\partial t - \partial E_t/\partial y \\ j_z &= (\partial B_x/\partial y - B_y/\partial x) - \partial E_z/\partial t - \partial E_t/\partial z \end{aligned}\right\} \Rightarrow \quad \mathrm{rot}\ \mathbf{B} - \partial\mathbf{E}/\partial t - \mathrm{grad}\ E_t = \mathbf{J} \qquad (1.4)'$$

$$0 = \partial B_x/\partial x + \partial B_y/\partial y + \partial B_z/\partial z \quad \Rightarrow \quad \mathrm{div}\ \mathbf{B} = 0 \qquad (1.2)$$

$$\left.\begin{aligned} 0 &= \partial B_x/\partial t + (\partial E_y/\partial z - \partial E_z/\partial y) \\ 0 &= \partial B_y/\partial t + (\partial E_z/\partial x - \partial E_x/\partial z) \\ 0 &= \partial B_z/\partial t + (\partial E_x/\partial y - \partial E_y/\partial x) \end{aligned}\right\} \Rightarrow \quad \mathrm{rot}\ \mathbf{E} + \partial\mathbf{B}/\partial t = 0 \qquad (1.1)$$

and from the equation $F = \tilde{D} \cdot \tilde{A}$, we may rewrite as

$$\begin{pmatrix} E_t+iB_t+E_x+iB_x & E_y+iB_y+iE_z-B_z \\ E_y+iB_y-iE_z+B_z & E_t+iB_t+E_x-iB_x \end{pmatrix} = \begin{pmatrix} \partial/\partial t-\partial/\partial x & -\partial/\partial y-i\partial/\partial z \\ -\partial/\partial y+i\partial/\partial z & \partial/\partial t+\partial/\partial x \end{pmatrix} \begin{pmatrix} \phi-A_x & -A_y-iA_z \\ -A_y+iA_z & \phi+A_x \end{pmatrix}$$

,i.e.,

$$E_t = \partial\phi/\partial t+\partial A_x/\partial x+\partial A_y/\partial y+\partial A_z/\partial z \quad \Rightarrow \quad E_t = \partial\phi/\partial t+\mathrm{div}\ \mathbf{A} \qquad (1.6)_t$$

$$\left.\begin{array}{l} E_x=-\partial A_x/\partial t-\partial\phi/\partial x \\ E_y=-\partial A_y/\partial t-\partial\phi/\partial y \\ E_z=-\partial A_z/\partial t-\partial\phi/\partial z \end{array}\right\} \quad \Rightarrow \quad \mathbf{E}=-\mathrm{grad}\phi-\partial\mathbf{A}/\partial t \qquad (1.6)$$

$$B_t = 0$$

$$\left.\begin{array}{l} B_x=\partial A_y/\partial z-\partial A_z/\partial y \\ B_y=\partial A_z/\partial x-\partial A_x/\partial z \\ B_z=\partial A_x/\partial y-\partial A_y/\partial x \end{array}\right\} \quad \Rightarrow \quad \mathbf{B}=\mathrm{rot}\ \mathbf{A} \qquad (1.5)$$

(where the B_t is the fourth component of the magnetic field)

and from the equation $\tilde{J} = D \cdot \tilde{D} \cdot \tilde{A}$, we may rewrite as

$$\begin{pmatrix} \rho-j_x & -j_y-ij_z \\ -j_y+ij_z & \rho+j_x \end{pmatrix} = \begin{pmatrix} \partial/\partial t+\partial/\partial x & \partial/\partial y+i\partial/\partial z \\ \partial/\partial y-i\partial/\partial z & \partial/\partial t-\partial/\partial x \end{pmatrix} \begin{pmatrix} \partial/\partial t-\partial/\partial x & -\partial/\partial y-i\partial/\partial z \\ -\partial/\partial y+i\partial/\partial z & \partial/\partial t+\partial/\partial x \end{pmatrix} \begin{pmatrix} \phi-A_x & -A_y-iA_z \\ -A_y+iA_z & \phi+A_x \end{pmatrix}$$

,i.e.,

$$\rho=\partial^2\phi/\partial t^2-(\partial^2\phi/\partial x^2+\partial^2\phi/\partial y^2+\partial^2\phi/\partial z^2) \quad \Rightarrow \quad \partial^2\phi/\partial t^2-\nabla^2\phi=\rho \qquad (1.7)$$

$$\left.\begin{array}{l} j_x=\partial^2 A_x/\partial t^2-(\partial^2 A_x/\partial x^2+\partial^2 A_x/\partial y^2+\partial^2 A_x/\partial z^2) \\ j_y=\partial^2 A_y/\partial t^2-(\partial^2 A_y/\partial x^2+\partial^2 A_y/\partial y^2+\partial^2 A_y/\partial z^2) \\ jz=\partial^2 A_z/\partial t^2-(\partial^2 A_z/\partial x^2+\partial^2 A_z/\partial y^2+\partial^2 A_z/\partial z^2) \end{array}\right\} \Rightarrow \quad \partial^2\mathbf{A}/\partial t^2-\nabla^2\mathbf{A}=\mathbf{J} \qquad (1.8)$$

and from the equation $0 = \tilde{D} \cdot D \cdot F$, we can get eqs. (1.9), (1.10) and $(1.10)_t$ in the same way as above.

q.e.d.

§4 The Yang-Mills functional

We take the 2×2 complex matrix $A = \begin{pmatrix} a_1+a_2 & a_3+ia_4 \\ a_3-ia_4 & a_1-a_2 \end{pmatrix}$

Let $\tilde{A}$ be a cofactor matrix of A and let $\frac{1}{2}(A+\tilde{A})$, $\frac{1}{2}(A-\tilde{A})$ be real part and imaginary part of A respectively.

,i.e.,

$$\tilde{A} = \begin{pmatrix} a_1 - a_2 & -a_3 - ia_4 \\ -a_3 + ia_4 & a_1 + a_2 \end{pmatrix}$$

$$\frac{1}{2}(A + \tilde{A}) = \begin{pmatrix} a_1 & 0 \\ 0 & a_1 \end{pmatrix} \quad , \quad \frac{1}{2}(A - \tilde{A}) = \begin{pmatrix} a_2 & a_3 + ia_4 \\ a_3 - ia_4 & -a_2 \end{pmatrix}$$

and $(A)_i = a_i$ $(i = 1 \sim 4)$

we consider the u(1)-principal fiber bundle P(M, u(1)) on the Minkowski space M and let the connection form be

$$A = \phi dt - A_x dx - A_y dy - A_z dz.$$

and then the curvature form is

$$\begin{aligned} F = \; & \underline{E_1 dt \wedge dt} + \tfrac{1}{2} E_x dt \wedge dx + \tfrac{1}{2} E_y dt \wedge dy + \tfrac{1}{2} E_z dt \wedge dz \\ & - \tfrac{1}{2} E_x dx \wedge dt + \underline{E_2 dx \wedge dx} - \tfrac{1}{2} B_z dx \wedge dy + \tfrac{1}{2} B_y dx \wedge dz \\ & - \tfrac{1}{2} E_y dy \wedge dt + \tfrac{1}{2} B_z dy \wedge dx + \underline{E_3 dy \wedge dy} - \tfrac{1}{2} B_x dy \wedge dz \\ & - \tfrac{1}{2} E_z dz \wedge dt - \tfrac{1}{2} B_y dz \wedge dx + \tfrac{1}{2} B_x dz \wedge dy + \underline{E_4 dz \wedge dz} \end{aligned}$$

(where the underlined parts are null)

$$\begin{aligned} = \; & \tfrac{1}{2}(E_x + iB_x)(dt \wedge dx + idy \wedge dz) + \tfrac{1}{2}(E_y + iB_y)(dt \wedge dy + idz \wedge dx) + \tfrac{1}{2}(E_z + iB_z)(dt \wedge dz + idx \wedge dy) \\ & + \tfrac{1}{2}(E_x - iB_x)(dt \wedge dx - idy \wedge dz) + \tfrac{1}{2}(E_y - iB_y)(dt \wedge dy - idz \wedge dx) + \tfrac{1}{2}(E_z - iB_z)(dt \wedge dz - idx \wedge dy) \end{aligned}$$

The E_x, E_y, E_z, B_x, B_y and B_z above are the space-components of the curvature.
and

$$E_x = -\partial A_x/\partial t - \partial\phi/\partial x, \qquad B_x = -\partial A_y/\partial z - \partial A_z/\partial y,$$

$$E_y = -\partial A_y/\partial t - \partial\phi/\partial y, \qquad B_y = -\partial A_z/\partial x - \partial A_x/\partial z,$$

$$E_z = -\partial A_z/\partial t - \partial\phi/\partial z, \qquad B_z = -\partial A_x/\partial y - \partial A_y/\partial x,$$

$$E_1 = \partial\phi/\partial t, \quad E_2 = \partial A_x/\partial x, \quad E_3 = \partial A_y/\partial y, \quad E_4 = \partial A_z/\partial z,$$

We define the time-component of the curvature as

$$E_t = \partial\phi/\partial t + \partial A_x/\partial x + \partial A_y/\partial y + \partial A_z/\partial z,$$
$$B_t = 0.$$

Let the connection matrix and the curvature matrix be

$$\tilde{A} = \begin{pmatrix} \phi - A_x & -A_y - iA_z \\ -A_y + iA_z & \phi + A_x \end{pmatrix} \quad , \quad F = \begin{pmatrix} E_t + iB_t + E_x + iB_x & E_y + iB_y + iE_z - B_z \\ E_y + iB_y - iE_z + B_z & E_t + iB_t - E_x - iB_x \end{pmatrix},$$

We define the norm of curvature as

$$|F| = (\tilde{F} \cdot F)_1 = (E_t + iB_t)^2 - (E_x + iB_x)^2 - (E_y + iB_y)^2 - (E_z + iB_z)^2$$

(the determinant of the curvature matrix F)

and the Yang-Mills functional as

$$YM : A \to \int_M |F|\, dv = \int_M (\tilde{F} \cdot F)_1 dv.$$

Let A_t be a curve in connection space such that $A_0 = A$ and $a_t = A_t - A$ and $\alpha = \frac{d}{dt}\alpha_t(0)$ (with compact support)

then

$$\frac{d}{dt} F(A_t)(0) = \frac{d}{dt}(F(A) + D \cdot a_t)(0) = D \cdot \alpha.$$

The following theorem holds.

Theorem 3.

$$\frac{d}{dt} YM(A)(0) = \int_M (DF \cdot \alpha)_1 dv$$

Proof

$$\frac{d}{dt} YM(A_t)\ (0) = \frac{d}{dt} \int_M (\tilde{F} \cdot F)_1 dv$$

$$= \int_M ((\widetilde{\tilde{D} \cdot \tilde{\alpha}}) \cdot F + \tilde{F} \cdot (\tilde{D} \cdot \tilde{\alpha}))_1 dv$$

$$= \int_M (\tilde{F} \cdot (\tilde{D} \cdot \tilde{\alpha}))_1 + (\tilde{F} \cdot (\tilde{D} \cdot \tilde{\alpha}))_1 dv$$

$$= 2\int_M (\tilde{F} \cdot (\tilde{D} \cdot \tilde{\alpha}))_1 dv$$

$$= 2\int_M ((D \cdot F) \cdot \alpha)_1) dv$$

$\because$ by the proposition 5, 6 below

q.e.d.

Corollary 4 .

The equation of the connection (or potential) matrix A

$$D \cdot F\ (A) = 0$$

is the condition of the Yang-Mills connection.

Proof

We can suppose that the α (in the above theorem) has the small compact support at any point . then we have

$$((D \cdot F) \cdot \alpha)_1 = 0\,.$$

Specially, we take α's as follows :

when $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then $(D \cdot F)_1 = 0$

holds.

when $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ then $(D \cdot F)_2 = 0$

holds.

when $\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ then $(D \cdot F)_3 = 0$

holds.

when $\alpha = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ then $(D \cdot F)_4 = 0$

holds.

q. e. d.

Proposition 5.

We take $A = \begin{pmatrix} a_1 + a_2 & a_3 + ia_4 \\ a_3 - ia_4 & a_1 - a_2 \end{pmatrix}$,

and let the matrix elements a_1, a_2, a_3, a_4 be the c^∞-function with compact support.

Then

$$\int_M (D \cdot A)_1 \, d\, v = 0$$

holds.

Proof.

$$(D \cdot A)_1 = \left(\begin{pmatrix} \partial/\partial x^1 + \partial/\partial x^2 & \partial/\partial x^3 + i\partial/\partial x^4 \\ \partial/\partial x^3 - i\partial/\partial x^4 & \partial/\partial x^1 - \partial/\partial x^2 \end{pmatrix} \begin{pmatrix} a_1 + a_2 & a_3 + ia_4 \\ a_3 - ia_4 & a_1 - a_2 \end{pmatrix} \right)_1$$

$$= \partial a_1/\partial x^1 + \partial a_2/\partial x^2 + \partial a_3/\partial x^3 + \partial a_4/\partial x^4$$

therehore, by the Green's theorem

$$\int_M (D \cdot A)_1 \, dv = 0$$

holds.

q.e.d.

Let

$$A = \begin{pmatrix} a_1 + a_2 & a_3 + ia_4 \\ a_3 - ia_4 & a_1 - a_2 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 + b_2 & b_3 + ib_4 \\ b_3 - ib_4 & b_1 - b_2 \end{pmatrix},$$

then the following proposition holds.

Proposition 6.

$$(D \cdot (A \cdot B))_1 = ((D \cdot A) \cdot B)_1 + (\tilde{A} \cdot (\tilde{D} \cdot \tilde{B}))_1$$

Proof.
we calculate $(D \cdot (A \cdot B))_1$ as follows :

$$A \cdot B = \begin{pmatrix} (a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4) & (a_1b_3 + ia_2b_4 + a_3b_1 - ia_4b_2) \\ \quad + (a_1b_2 + a_2b_1 - ia_3b_4 + ia_4b_3) & \quad + i(a_1b_4 - ia_2b_3 + ia_3b_2 + a_4b_1) \\ (a_1b_3 + ia_2b_4 + a_3b_1 - ia_4b_2) & (a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4) \\ \quad - i(a_1b_4 - ia_2b_3 + ia_3b_2 + a_4b_1) & \quad - (a_1b_2 + a_2b_1 - ia_3b_4 + ia_4b_3) \end{pmatrix}$$

$$\begin{aligned} &(D \cdot (A \cdot B))_1 \\ &= \partial(a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4)/\partial x^1 + \partial(a_1b_2 + a_2b_1 - ia_3b_4 + ia_4b_3)/\partial x^2 \\ &\quad + \partial(a_1b_3 + ia_2b_4 + a_3b_1 - ia_4b_2)/\partial x^3 + \partial(a_1b_4 - ia_2b_3 + ia_3b_2 + a_4b_1)/\partial x^4 \end{aligned}$$

we calculate $((D \cdot A) \cdot B)_1$ as follows :

$$D \cdot A = \begin{pmatrix} (\partial a_1/\partial x^1 + \partial a_2/\partial x^2 + \partial a_3/\partial x^3 + \partial a_4/\partial x^4) & (\partial a_3/\partial x^1 + i\partial a_4/\partial x^2 + \partial a_1/\partial x^3 - i\partial a_2/\partial x^4) \\ + (\partial a_2/\partial x^1 + \partial a_1/\partial x^2 - i\partial a_4/\partial x^3 + i\partial a_3/\partial x^4) & + i(\partial a_4/\partial x^1 - \partial a_3/\partial x^2 + i\partial a_2/\partial x^3 + \partial a_1/\partial x^4) \\ (\partial a_3/\partial x^1 + i\partial a_4/\partial x^2 + \partial a_1/\partial x^3 - i\partial a_2/\partial x^4) & (\partial a_1/\partial x^1 + \partial a_2/\partial x^2 + \partial a_3/\partial x^3 + \partial a_4/\partial x^4) \\ - i(\partial a_4/\partial x^1 - \partial a_3/\partial x^2 + i\partial a_2/\partial x^3 + \partial a_1/\partial x^4) & - (\partial a_2/\partial x^1 + \partial a_1/\partial x^2 - i\partial a_4/\partial x^3 + i\partial a_3/\partial x^4) \end{pmatrix},$$

$((D \cdot A) \cdot B)_1$

$$=(\partial a_1/\partial x^1+\partial a_2/\partial x^2+\partial a_3/\partial x^3+\partial a_4/\partial x^4)b_1+(\partial a_2/\partial x^1+\partial a_1/\partial x^2-\partial a_4/\partial x^3+i\partial a_3/\partial x^4)b_2$$
$$+(\partial a_3/\partial x^1+i\partial a_4/\partial x^2-\partial a_1/\partial x^3-i\partial a_2/\partial x^4)b_3+(\partial a_4/\partial x^1-\partial a_3/\partial x^2+i\partial a_2/\partial x^3+\partial a_1/\partial x^4)b_4.$$

we calculate $(\tilde{A}\cdot(\tilde{D}\cdot\tilde{B}))_1$ as follows :

$$\tilde{D}\cdot\tilde{B}=\begin{pmatrix} (\partial b_1/\partial x^1+\partial b_2/\partial x^2+\partial b_3/\partial x^3+\partial b_4/\partial x^4) & -(\partial b_3/\partial x^1-i\partial b_4/\partial x^2+\partial b_1/\partial x^3-i\partial b_2/\partial x^4) \\ -(\partial b_2/\partial x^1+\partial b_1/\partial x^2+i\partial b_4/\partial x^3-i\partial b_3/\partial x^4) & -i(\partial b_4/\partial x^1+\partial b_3/\partial x^2-i\partial b_2/\partial x^3+\partial b_1/\partial x^4) \\ -(\partial b_3/\partial x^1-i\partial b_4/\partial x^2+\partial b_1/\partial x^3+i\partial b_2/\partial x^4) & (\partial b_1/\partial x^1+\partial b_2/\partial x^2+\partial b_3/\partial x^3+\partial b_4/\partial x^4) \\ +i(\partial b_4/\partial x^1+\partial b_3/\partial x^2-i\partial b_2/\partial x^3+\partial b_1/\partial x^4) & +(\partial b_2/\partial x^1+\partial b_1/\partial x^2+i\partial b_4/\partial x^3-i\partial b_3/\partial x^4) \end{pmatrix},$$

$$(\tilde{A}\cdot(\tilde{D}\cdot\tilde{B}))_1$$
$$=\ a_1(\partial b_1/\partial x^1+\partial b_2/\partial x^2+\partial b_3/\partial x^3+\partial b_4/\partial x^4)+a_2(\partial b_2/\partial x^1+\partial b_1/\partial x^2+\partial b_4/\partial x^3-i\partial b_3/\partial x^4)$$
$$+a_3(\partial b_3/\partial x^1-i\partial b_4/\partial x^2+\partial b_1/\partial x^3+i\partial b_2/\partial x^4)+a_4(\partial b_4/\partial x^1+i\partial b_3/\partial x^2-i\partial b_2/\partial x^3+\partial b_1/\partial x^4).$$

therefore

$$((D\cdot A)\cdot B)_1+(\tilde{A}\cdot(\tilde{D}\cdot\tilde{B}))_1$$
$$=\ \{\partial a_1/\partial x^1 b_1+\partial a_2/\partial x^1 b_2+\partial a_3/\partial x^1 b_3+\partial a_4/\partial x^1 b_4\}+\{a_1\partial b_1/\partial x^1+a_2\partial b_2/\partial x^1+a_3\partial b_3/\partial x^1+a_4\partial b_4/\partial x^1\}$$
$$+\{\partial a_1/\partial x^2 b_2+\partial a_2/\partial x^2 b_1-i\partial a_3/\partial x^2 b_4+i\partial a_4/\partial x^2 b_3\}+\{a_1\partial b_2/\partial x^2+a_2\partial b_1/\partial x^2-ia_3\partial b_4/\partial x^2+ia_4\partial b_3/\partial x^2\}$$
$$+\{\partial a_1/\partial x^3 b_3+i\partial a_2/\partial x^3 b_4+\partial a_3/\partial x^3 b_1-i\partial a_4/\partial x^3 b_2\}+\{a_1\partial b_3/\partial x^3+ia_2\partial b_4/\partial x^3+a_3\partial b_1/\partial x^3-ia_4\partial b_2/\partial x^3\}$$
$$+\{\partial a_4/\partial x^3 b_4-i\partial a_2/\partial x^4 b_3+\partial a_3/\partial x^4 b_1-i\partial a_4/\partial x^4 b_2\}+\{a_1\partial b_4/\partial x^3-ia_2\partial b_3/\partial x^4+ia_3\partial b_2/\partial x^4+a_4\partial b_1/\partial x^4\}$$

q.e.d.

§ 5 The complexification of the electromagnetic field

We take complex potential matrix as

$$\tilde{\mathbf{A}}=\begin{pmatrix} \mathbf{\Phi}-\mathbf{A}_x & -\mathbf{A}_y-i\mathbf{A}_z \\ -\mathbf{A}_y+i\mathbf{A}_z & \mathbf{\Phi}+\mathbf{A}_x \end{pmatrix},$$

and matrix elements as $\mathbf{\Phi}=\phi+i\phi'$, $\mathbf{A}_x=A_x+iA_x'$, $\mathbf{A}_y=A_y+iA_y'$, $\mathbf{A}_z=A_z+iA_z'$.

Then the electromagnetic field which is derived from the complex potential $\tilde{A}$ is as follows :

$$\begin{pmatrix} \mathbf{E}_t+\mathbf{E}_x & \mathbf{E}_y+i\mathbf{E}_z \\ \mathbf{E}_y-i\mathbf{E}_z & \mathbf{E}_t-\mathbf{E}_x \end{pmatrix}=\begin{pmatrix} \partial/\partial t-\partial/\partial x & -\partial/\partial y-i\partial/\partial z \\ -\partial/\partial y+i\partial/\partial z & \partial/\partial t+\partial/\partial x \end{pmatrix}\begin{pmatrix} \mathbf{\Phi}-\mathbf{A}_x & -\mathbf{A}_y-i\mathbf{A}_z \\ -\mathbf{A}_y+i\mathbf{A}_z & \mathbf{\Phi}+\mathbf{A}_x \end{pmatrix}$$

$$\begin{aligned}
\mathbf{E}_t &= \partial\Phi/\partial t+\partial \mathbf{A}_x/\partial x+\partial \mathbf{A}_y/\partial y+\partial \mathbf{A}_z/\partial z && \Rightarrow \quad \mathbf{E}_t=\partial\Phi/\partial t+\mathrm{div}\ \mathsf{A}\\
\mathbf{E}_x &= -\partial \mathbf{A}_x/\partial t-\partial\Phi/\partial x+i(\partial \mathbf{A}_y/\partial z-\partial \mathbf{A}_z/\partial y) && \\
\mathbf{E}_y &= -\partial \mathbf{A}_y/\partial t-\partial\Phi/\partial y+i(\partial \mathbf{A}_z/\partial x-\partial \mathbf{A}_x/\partial z) && \Rightarrow \quad \mathsf{E}=-\mathrm{grad}\Phi-\partial \mathsf{A}/\partial t+i\ \mathrm{rot}\ \mathsf{A}\\
\mathbf{E}_z &= -\partial \mathbf{A}_z/\partial t-\partial\Phi/\partial z+i(\partial \mathbf{A}_x/\partial y-\partial \mathbf{A}_y/\partial z) &&
\end{aligned}$$

(where $\mathbf{E}_t=E_t+iB_t$, $\mathbf{E}_x=E_x+iB_x$, $\mathbf{E}_y=E_y+iB_y$, $\mathbf{E}_z=E_z+iB_z$) .

We take the real and imaginary parts of the above equations, then we obtain the following equations.

(the real part)

$$\begin{aligned}
E_t &= \partial\phi/\partial t+\partial A_x/\partial x+\partial A_y/\partial y+\partial A_z/\partial z && \Rightarrow \quad E_t=\partial\phi/\partial t+\mathrm{div}\ \mathsf{A} && (1.6)_t\\
E_x &= -\partial A_x/\partial t-\partial\phi/\partial x-(\partial A_y'/\partial z-\partial A_z'/\partial y) && \\
E_y &= -\partial A_y/\partial t-\partial\phi/\partial y-(\partial A_z'/\partial x-\partial A_x'/\partial z) && \Rightarrow \quad \mathsf{E}=-\mathrm{grad}\phi-\partial \mathsf{A}/\partial t-\underline{\mathrm{rot}\ \mathsf{A}'} && (1.6)'\\
E_z &= -\partial A_z/\partial t-\partial\phi/\partial z-(\partial A_x'/\partial y-\partial A_y'/\partial z) &&
\end{aligned}$$

(the imaginary part)

$$\begin{aligned}
B_t &= \partial\phi'/\partial t+\partial A_x'/\partial x+\partial A_y'/\partial y+\partial A_z'/\partial z && \Rightarrow \quad B_t=\underline{\partial\phi'/\partial t}+\underline{\mathrm{div}\ \mathsf{A}'} && (1.5)_t\\
B_x &= -\partial A_x'/\partial t-\partial\phi'/\partial x+(\partial A_y/\partial z-\partial A_z/\partial y) && \\
B_y &= -\partial A_y'/\partial t-\partial\phi'/\partial y+(\partial A_z/\partial x-\partial A_x/\partial z) && \Rightarrow \quad \mathsf{B}=-\underline{\mathrm{grad}\phi'}-\underline{\partial \mathsf{A}'/\partial t}+\mathrm{rot}\ \mathsf{A} && (1.5)'\\
B_z &= -\partial A_z'/\partial t-\partial\phi'/\partial z+(\partial A_x/\partial y-\partial A_y/\partial z) &&
\end{aligned}$$

(where the underlined parts are the ones which are obtained from the imaginary part of the potential) .

We assume that the complex scalar and vector potential satisfy the wave equations (or condition of the Yang-Mills connection)

,i.e.,

$$\begin{aligned}
\partial^2\phi / \partial t^2-\nabla^2\phi &= \rho\\
\partial^2 A_x / \partial t^2-\nabla^2 A_x &= J_x\\
\partial^2 A_y / \partial t^2-\nabla^2 A_y &= J_y\\
\partial^2 A_z / \partial t^2-\nabla^2 A_z &= J_z
\end{aligned}$$

(where ρ, J_x, J_y, J_z be complex charge and current.)

and take the complex current matrix as

$$\tilde{J}=\begin{pmatrix} \rho-J_x & -J_y-iJ_z\\ -J_y+iJ_z & \rho+J_x\end{pmatrix}$$

and the matrix element as $\rho=\rho+i\rho'$, $J_x=j_y+i\ j_y'$, $J_z=j_z+i\ j_z'$.

Then the equations which are satisfied by the above electromagnetic field is as follows :

$$\begin{pmatrix} \rho-J_x & -J_y+iJ_z\\ -J_y-iJ_z & \rho+J_x\end{pmatrix}=\begin{pmatrix} \partial/\partial t+\partial/\partial x & \partial/\partial y+i\partial/\partial z\\ \partial/\partial y-i\partial/\partial z & \partial/\partial t-\partial/\partial z\end{pmatrix}\begin{pmatrix} \mathbf{E}_t+\mathbf{E}_x & \mathbf{E}_y+i\mathbf{E}_z\\ \mathbf{E}_y-i\mathbf{E}_z & \mathbf{E}_t-\mathbf{E}_x\end{pmatrix}$$

$$\rho = \partial \mathbf{E}_t/\partial t + (\partial \mathbf{E}_x/\partial x + \partial \mathbf{E}_y/\partial y + \partial \mathbf{E}_z/\partial z) \quad \Rightarrow \quad \mathrm{div}\mathbf{E} + \partial \mathbf{E}_t/\partial t = \rho$$

$$\left.\begin{aligned} J_x &= -\partial \mathbf{E}_x/\partial t - \partial \mathbf{E}_t/\partial x - i(\partial \mathbf{E}_y/\partial z - \partial \mathbf{E}_z/\partial y) \\ J_y &= -\partial \mathbf{E}_y/\partial t - \partial \mathbf{E}_t/\partial y - i(\partial \mathbf{E}_z/\partial x - \partial \mathbf{E}_x/\partial z) \\ J_z &= -\partial \mathbf{E}_z/\partial t - \partial \mathbf{E}_t/\partial z - i(\partial \mathbf{E}_x/\partial y - \partial \mathbf{E}_y/\partial x) \end{aligned}\right\} \Rightarrow \quad -i\ \mathrm{rot}\ \mathbf{E} - \partial \mathbf{E}/\partial t - \mathrm{grad}\mathbf{E}_t = \mathbf{J}$$

We take the real and imaginary parts of the above equations, then we obtain the following equations.

(the real part)

$$\rho = \partial E_t/\partial t + \partial E_x/\partial x + \partial E_y/\partial y + \partial E_z/\partial z \quad \Rightarrow \quad \mathrm{div}\ \mathbf{E} + \partial E_t/\partial t = \rho \qquad (1.3)'$$

$$\left.\begin{aligned} j_x &= -\partial E_x/\partial t - \partial E_t/\partial x + (\partial B_y/\partial z - \partial B_z/\partial y) \\ j_y &= -\partial E_y/\partial t - \partial E_t/\partial y + (\partial B_z/\partial x - \partial B_x/\partial z) \\ j_z &= -\partial E_z/\partial t - \partial E_t/\partial z + (\partial B_x/\partial y - \partial B_y/\partial x) \end{aligned}\right\} \Rightarrow \quad \mathrm{rot}\ \mathbf{B} - \partial \mathbf{E}/\partial t - \mathrm{grad}\ E_t = \mathbf{J} \qquad (1.4)'$$

(the imaginary part)

$$\rho' = \partial B_t/\partial t + \partial B_x/\partial x + \partial B_y/\partial y + \partial B_z/\partial z \quad \Rightarrow \quad \mathrm{div}\ \mathbf{B} + \underline{\partial B_t/\partial t = \rho'} \qquad (1.2)'$$

$$\left.\begin{aligned} j_x' &= -\partial B_x/\partial t - \partial B_t/\partial x - (\partial E_y/\partial z - \partial E_z/\partial y) \\ j_y' &= -\partial B_y/\partial t - \partial B_t/\partial y - (\partial E_z/\partial x - \partial E_x/\partial z) \\ j_z' &= -\partial B_z/\partial t - \partial B_t/\partial z - (\partial E_x/\partial y - \partial E_y/\partial x) \end{aligned}\right\} \Rightarrow \quad \mathrm{rot}\ \mathbf{E} + \partial \mathbf{B}/\partial t + \underline{\mathrm{grad}\ B_t} = -\underline{\mathbf{J}'} \qquad (1.1)'$$

(where the underlined parts are the ones which are lacking in the modified Maxwell equation)

§ 6 The transformation for the special theory of relativity

At last, we calculate the transformation for the special theory of relativity.

For example, we consider the Lorentz transformation on the t-x coordinate.

and let the Lorentz transformation(t − x) be

$$\begin{aligned} ct' &= \gamma(ct - \beta x)\ , \\ x' &= \gamma(x - \beta ct)\ , \\ y' &= y\ , \\ z' &= z\ . \end{aligned}$$

$$(\text{where } \beta = V/c\ ,\quad \gamma = (1 - V^2/c^2)^{-1/2})$$

Then the differential operators D, $\tilde{D}$ are transformed as follows :

$$\begin{pmatrix} \partial/\partial ct' + \partial/\partial x' & \partial/y' + i\partial/\partial z' \\ \partial/\partial y' - i\partial/\partial z' & \partial/ct' - \partial/\partial x' \end{pmatrix} = \begin{pmatrix} \gamma(1+\beta)(\partial/\partial ct + \partial/\partial x) & \partial/\partial y + i\partial/\partial z \\ \partial/\partial y - i\partial/\partial z & \gamma(1-\beta)(\partial/\partial ct - \partial/\partial x) \end{pmatrix}$$

$$\begin{pmatrix} \partial/\partial ct' - \partial/\partial x' & \partial/y' - i\partial/\partial z' \\ -\partial/\partial y' + i\partial/\partial z' & \partial/ct' + \partial/\partial x' \end{pmatrix} = \begin{pmatrix} \gamma(1-\beta)(\partial/\partial ct - \partial/\partial x) & -\partial/\partial y - i\partial/\partial z \\ -\partial/\partial y + i\partial/\partial z & \gamma(1+\beta)(\partial/\partial ct + \partial/\partial x) \end{pmatrix}$$

Proof

$$\partial/\partial ct' = \partial/\partial ct + \partial ct/\partial ct' + \partial/\partial x \cdot \partial x/\partial ct'$$
$$= \gamma(\partial/ct + \beta \cdot \partial/\partial x),$$
$$\partial/\partial x' = \partial/\partial x \cdot \partial x/\partial x' + \partial/\partial ct \cdot \partial ct/\partial x'$$
$$= \gamma(\partial/x + \beta \cdot \partial/\partial ct),$$
$$\partial/\partial y' = \partial/\partial y,$$
$$\partial/\partial z' = \partial/\partial z.$$

q.e.d.

And potential A is transformed as follows:

$$\boxed{\begin{pmatrix} 1/c \cdot \Phi' - A'_x & -A'_y - i A'_z \\ -A'_y + i A'_z & 1/c \cdot \Phi' + A'_x \end{pmatrix} = \begin{pmatrix} \gamma(1+\beta)(1/c \cdot \Phi - A_x) & -A_y - i A_z \\ -A_y + i A_z & \gamma(1-\beta)(1/c \cdot \Phi + A_x) \end{pmatrix}}$$

Proof

$$1/c \cdot \Phi' = \gamma\ (1/c \cdot \Phi - \beta \cdot A_x),$$
$$A'_x = \gamma\ (A_x - \beta/c \cdot \Phi),$$
$$A'_y = A_y,$$
$$A'_z = A_z.$$

q.e.d.

Therefore, the transformation of the field F is calculated as follows:

$$\begin{pmatrix} E'_t + E'_x & E'_y + i E'_z \\ E'_y - E'_z & E'_t - E'_x \end{pmatrix}$$
$$= \begin{pmatrix} \partial/\partial ct' - \partial/\partial x' & -\partial/\partial y' - i\partial/\partial z' \\ -\partial/\partial y' + i\partial/\partial z' & \partial/\partial ct' + \partial/\partial x' \end{pmatrix} \begin{pmatrix} 1/c \cdot \Phi' - A'_x & -A'_y - i A'_z \\ -A'_y + i A'_z & 1/c \cdot \Phi' + A'_x \end{pmatrix}$$
$$= \begin{pmatrix} \gamma(1-\beta)(\partial/\partial ct - \partial/\partial x) & -\partial/\partial y - i\partial/\partial z \\ -\partial/\partial y + i\partial/\partial z & \gamma(1+\beta)(\partial/\partial ct + \partial/\partial x) \end{pmatrix} \begin{pmatrix} \gamma(1+\beta)(1/c \cdot \Phi - A_x) & -A_y - i A_z \\ -A_y + i A_z & \gamma(1-\beta)(1/c \cdot \Phi + A_x) \end{pmatrix}$$
$$= \begin{pmatrix} E_t + E_x & \gamma(1-\beta)(E_y + i E_z) \\ \gamma(1+\beta)(E_y - i E_z) & E_t - E_x \end{pmatrix}.$$

Therefore, for the time-component E_t, there are no effects by the special theory of relativity and also the norm of the field (or curvature) $|F|$ is invariant.

Moreover, the transformation of the charge and current $\tilde{J}$ is calculated as follows:

$$\begin{pmatrix} c\rho' - J'_x & -J'_y - i J_z \\ -J'_y + i J_z & c\rho' + J'_x \end{pmatrix}$$
$$= \begin{pmatrix} \partial/\partial ct' + \partial/\partial x' & \partial/\partial y' + i\partial/\partial z' \\ \partial/\partial y' - i\partial/\partial z' & \partial/\partial ct' - \partial/\partial x' \end{pmatrix} \begin{pmatrix} E'_t + E'_x & E'_y + i E'_z \\ E'_y - E'_z & E'_t - E'_x \end{pmatrix}$$

$$= \begin{pmatrix} \gamma(1+\beta)(\partial/\partial ct+\partial/\partial x) & \partial/\partial y+i\partial/\partial z \\ \partial/\partial y-i\partial/\partial z & \gamma(1-\beta)(\partial/\partial ct-\partial/\partial x) \end{pmatrix} \begin{pmatrix} E_t+E_x & \gamma(1-\beta)(E_y+iE_z) \\ \gamma(1+\beta)(E_y-iE_z) & E_t-E_x \end{pmatrix}$$

$$= \begin{pmatrix} \gamma(1+\beta)(c\rho-J_x) & -J_y-iJ_z \\ -J_y+iJ_z & \gamma(1-\beta)(c\rho+J_x) \end{pmatrix}$$

This transformation is of the same form as the one of the potential $\tilde{A}$.

Some questions :

1. For the equations (in §2)

   $\mathrm{div}\mathbf{E}+\partial E_t/\partial t=\rho$,

   $\mathrm{rot}\mathbf{B}-\partial \mathbf{E}/\partial t-\mathrm{grad}E_t=\mathbf{J}$,

   can we detect the fourth component E_t of the electric field ?
2. For the equations (in §5)

   $\mathrm{div}\mathbf{B}+\partial B_t/\partial t=\rho'$,

   $-\mathrm{rot}\mathbf{E}-\partial \mathbf{B}/\partial t-\mathrm{grad}B_t=\mathbf{J}'$,

   is the magnetic charge ρ', so called, a magnetic monopole ?

   And can we detect the fourth component B_t of the magnetic field ?

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