"Pseudo" - Fiber Bundle and Connection on It

(An extension of fiber and connection's concept)

Yoshio TAKEMOTO

日 本 文 理 大 学 紀 要
第20巻 第1号
平成4年2月

(Bulletin of Nippon Bunri University
Vol. 20, No. 1 (1992-Feb.))

"Pseudo" - Fiber Bundle and Connection on It*

(An extension of fiber and connection's concept)

Yoshio TAKEMOTO**

College of Liberal Arts & Sciences,
Nippon Bunri University

Abstract

When we formally change the base space of the BPST-instanton, i. e., $\mathbb{R}^4$ (or S^4) to the Minkowski (or anti de Sitter space) and then we encounter two problems (1), (2) as follows : The instanton on the Minkowski space is

$$A(u) = \frac{1}{1+|u|} \operatorname{Im} \tilde{u} du \in SL(2, \mathbb{C})$$

and

(1) to construct an instanton by means of concrete method, and

(2) to dissolve the singularity at the points which satisfy the relation $1 + |u| = 0$.

For the problem (1), we construct the bundle for the Minkowski case and the instanton as the connection of this bundle.

For the problem (2), we find that the singularities occur at infinity in anti-de Sitter space, and hence we extend the concept of the bundle and define a new connection on it. And then we connect the two and over principal fiber bundle and each connection smoothly.

In this Minkowski case, the extended fiber bundle is covered by six principal fiber bundles whose base space is an anti-de Sitter space or a de Sitter space. Then, the correspondences of singularities between the base spaces are as follows :

A point at infinity (or an inner point) of the anti-de Sitter space correspond to an inner point (or a point at infinity) of the de Sitter space, and vice versa.

We discuss the above contents also for the complex case because the argument becomes simpler and further the Euclidean case and the Minkowski case are realized by restricting on each subspace.

* Received Oct 29, 1991

** Associate Professor, College of Liberal Arts & Sciences

§ 1. Introduction

We state an outline about the Hopf fiber bundle in this section.

Let $\mathbb{H}$ be a quaternion and the following relation "$\sim$" is an equivalent relation , i.e.,

$$\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} \sim \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} \Leftrightarrow \left\{ \begin{array}{c} \text{there is a number } \lambda \in \mathbb{H} \text{ which satisfy} \\ X_2 = X_1 \lambda \\ Y_2 = Y_1 \lambda \end{array} \right\}$$

Then we call the space $\mathbb{HP}_2 = (\mathbb{H}^2 - \{0\})/\sim$ the quaternion projective space. This space is isomorphic to the 4-dimentional sphere S^4.

Because the projective line $\mathbb{HP}(1)$ over $\mathbb{H}$'is a matrix expression of the projective space $\mathbb{HP}_2$ by the following mapping f:

,i.e.,

$$f: \begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} \xi & \eta \\ \bar{\eta} & 1-\xi \end{pmatrix} = \frac{1}{|X|^2 + |Y|^2} \begin{pmatrix} |X|^2 & X\bar{Y} \\ \bar{Y}X & |Y|^2 \end{pmatrix}, \text{ and}$$

$$\mathbb{HP}(1) = \{m \in \mathbf{M}(2, \mathbb{H}) / {}^t\bar{m} = m,\ m^2 = m,\ tr(m) = 1\}$$

$$= \left\{ m = \begin{pmatrix} \xi & \eta \\ \bar{\eta} & 1-\xi \end{pmatrix} / (1-2\xi)^2 + |2\eta|^2 = 1,\ \xi \in \mathbb{R},\ \eta \in \mathbb{H} \right\}$$

(where $\bar{\eta}$ and $|\eta|$ mean the conjugate number and the absolute value of η).

And this space is isomorphic to the sphere S^4 by the following mapping ϕ:

$$\phi: \begin{pmatrix} \xi & \eta \\ \bar{\eta} & 1-\xi \end{pmatrix} \to (z_0, Z) = (1-2\xi, 2\eta) \in \mathbb{R} \times \mathbb{H}.$$

Therefore we consider the following fiber bundle and the projection π:

$$\begin{array}{ccl} & \mathbb{H}^2 - \{0\} & \\ \pi = \phi \cdot f & \big\downarrow & \{\lambda \in \mathbb{H} / |\lambda| \neq 0\} \quad (\text{str. gr.}), \\ & \mathbb{HP}_2 \cong S^4 & \end{array}$$

and

$$\pi: \begin{pmatrix} X \\ Y \end{pmatrix} \to (z_0, Z) = \frac{1}{|X|^2 + |Y|^2} (|Y|^2 - |X|^2, 2X\bar{Y}).$$

Definition 1. (the Hopf fiber bundle)

We call the following fiber bundle

$$S^7 = \left\{ \begin{pmatrix} X \\ Y \end{pmatrix} \in \mathbb{H}^2 \,/\, |X|^2+|Y|^2=1 \right\}$$

$$\pi \Big\downarrow \quad \mathbb{H}_0 = \{\lambda \in \mathbb{H} \,/\, |\lambda|=1\}$$

$$S^4$$

the Hopf fiber bundle and the mapping π: $S_H{}^1 \to S^4$ the Hopf mapping.

Proposition 2.

There is a canonical connection on the Hopf fiber bundle ,i.e.,

$$\Theta = \mathrm{Im}\,(\bar{X}dX + \bar{Y}dY).$$

Proof

Let $\left\langle \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}, \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} \right\rangle = \bar{X}_1 Y_2 + \bar{Y}_1 Y_2$ be an inner prodect on the bundle space.

Then the tangent vectors $\begin{pmatrix} dX \\ dY \end{pmatrix}$ on the bundle space satisfy the following condition:

$$d \left\langle \begin{pmatrix} X \\ Y \end{pmatrix}, \begin{pmatrix} X \\ Y \end{pmatrix} \right\rangle = d\,(X\bar{X}+Y\bar{Y}) = 2\,\mathrm{Re}\,(X\bar{d}X+Y\bar{d}Y) = 0. \qquad \cdots (1)$$

And the horizontal condition of the above tangent vectors $\begin{pmatrix} dX \\ dY \end{pmatrix}$ is as follows:

$$\left\langle \begin{pmatrix} X \\ Y \end{pmatrix}, \begin{pmatrix} dX \\ dY \end{pmatrix} \right\rangle = \bar{X}dX+\bar{Y}dY = 0. \qquad \cdots (2)$$

Therefore the connection form Θ on the bundle space satisfy the following relation.

$$\left\langle \begin{pmatrix} X \\ Y \end{pmatrix}, \begin{pmatrix} dX \\ dY \end{pmatrix} - \begin{pmatrix} X \\ Y \end{pmatrix} \Theta \right\rangle = 0 \quad (\because \text{by } (2))$$

$$\therefore \quad \Theta = \bar{X}dX+\bar{Y}dY$$
$$= \mathrm{Im}\,(\bar{X}dX+\bar{Y}dY) \qquad (\because \text{by } (1))$$

q.e.d.

The base space $S^4 = \{(z_0, Z) \in \mathbb{R}\times\mathbb{H} \,/\, (z_0)^2 + |Z|^2 = 1\}$ is covered by two open sets, i. e.,

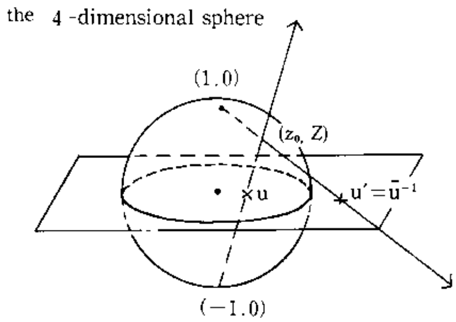
$S^4\,(z_0 \neq -1) = S^4 \cap \{z_0 \neq -1\}$,

$S^4\,(z_0 \neq 1) = S^4 \cap \{z_0 \neq 1\}$,

and stereographic projection ϕ_{-1} on $S^4(z_0 \neq -1)$ and ϕ_1 on $S^4(z_0 \neq 1)$ are as follows:

$$\phi_{-1}: (z_0, Z) \to u = \frac{Z}{z_0+1} = XY^{-1} \in \mathbb{H}(u),$$

(see figure 1.)



(figure 1)

$$\phi_1 : (z_0, Z) \to u' = \frac{\bar{Z}}{1-z_0} = \bar{Y}^{-1}\bar{X} \quad \in \mathbb{H}(u'),$$

(where X^{-1} and Y^{-1} are reciprocal number of X and Y respectively)

then the cross section μ_{-1} on $\mathbb{H}(u)$ and μ_1 on $\mathbb{H}(u')$ are as follows :

$$\mu_{-1}: u \to \frac{1}{(1+|u|^2)^{1/2}} \begin{pmatrix} u \\ 1 \end{pmatrix} \in S^7,$$

$$\mu_1 : u' \to \frac{1}{(1+|u'|^2)^{1/2}} \begin{pmatrix} 1 \\ \bar{u}' \end{pmatrix} \in S^7.$$

the connection form A on the base space is the pull back of Θ by the above cross section $\mu_i (i=\pm 1)$,i.e.,

$$\begin{aligned} A_{-1} &= (\mu_{-1}{}^*\Theta)\,(u) \\ &= \mathrm{Im}\, \frac{1}{1+|u|}\, \bar{u}du \qquad u \in \mathbb{H}(u), \\ A_1 &= (\mu_1{}^*\,\Theta)\,(u') \\ &= \mathrm{Im}\, \frac{1}{1+|u'|}\, u'd\bar{u}' \qquad u' \in \mathbb{H}(u'). \end{aligned}$$

We consider the extended Hoph fiber bundle in the next section.

The base space of this fiber bundle is 8-dimentional manifold and contains the 4-dimentional sphere (i. e., the base space. of the Hoph fiber bundle), de Sitter space and anti-de Sitter space.

As a preparation, we rewrite the quaternion (i.e. Euclidean space) and Minkowski space with a matrix style as follows:

$$\text{Let } \mathbb{R}^4 = \left\{ u = \begin{pmatrix} u_1+iu_2 & u_3+iu_4 \\ -u_3+iu_4 & u_1-iu_2 \end{pmatrix} \Big/ u_1,u_2,u_3,u_4 \in \mathbb{R} \right\} \text{ be an Euclidean space.}$$

then a quaternion $u_q = u_1 + iu_2 + ju_3 + ku_4$, a conjugate $\bar{u}_q$, an absolute value $|u_q|$ are correspond to a matrix u, a cofactor matrix $\tilde{u}$ and a square root of determinant $|u|$.

$$\text{Let } \mathbb{R}^{1,3} = \left\{ u = \begin{pmatrix} u_1+u_2 & u_3+iu_4 \\ u_3-iu_4 & u_1-u_2 \end{pmatrix} \Big/ u_1,u_2,u_3,u_4 \in \mathbb{R} \right\} \text{ be a Minkowski space.}$$

Then $\tilde{u}$ is a cofactor matrix $\begin{pmatrix} u_1-u_2 & -u_3-iu_4 \\ -u_3+iu_4 & u_1+u_2 \end{pmatrix}$ which corresponds to a conjugate. and $|u|$ is a determinant $(u_1)^2-(u_2)^2-(u_3)^2-(u_4)^2$ which corresponds to a square of absolute value.

§ 2. The extended Hopf fiber bundle

Definition 3. (the principal fiber bundle)

Let **M** be a manifold and **G** a Lie group. A principal fiber bundle over **M** with group **G** consists of a manifold **P** and an action of **G** on **P** satisfying the following conditions:
(1) **G** acts freely on **P** on the right: $(u, a) \in \mathbf{P}\times\mathbf{G} \rightarrow ua=\mathbf{R}_a u \in \mathbf{P}$;

(2) **M** is the quotient space of **P** by the equivalence relation induced by **G**, $\mathbf{M}=\mathbf{P}/\mathbf{G}$, and the canonical projection π: $\mathbf{P} \rightarrow \mathbf{M}$ is differentiable;

(3) **P** is locally trivial, that is, every point x of **M** has a neighborhood U such that π^{-1} (U) is isomorphic with $U\times\mathbf{G}$ in the sense that there is a diffeomorphism $\Psi(u) = (\pi(u), \Phi(u))$ where Φ is a mapping of π^{-1} (U) into **G** satisfying $\Phi(ua) = (\Phi(u))\, a$ for all $u \in \pi^{-1}$ (U) and $a \in \mathbf{G}$.

A principal fiber bundle will be denoted by **P** (**M**, **G**, π).

Let

$$M_2(\mathbb{C}) = \left\{ u=\begin{pmatrix} u_1+u_2 & u_3+iu_4 \\ u_3-iu_4 & u_1-u_2 \end{pmatrix} \;\middle|\; u_1,u_2,u_3,u_4 \in \mathbb{C} \right\},$$

and

$\tilde{u}=\begin{pmatrix} u_1-u_2 & -u_3-u_4 \\ -u_3+iu_4 & u_1+u_2 \end{pmatrix}$ is a cofactor matrix and $|u|=(u_1)^2-(u_2)^2-(u_3)^2-(u_4)^2$ is a determinant.

Then we consider the following mapping on $M_2(\mathbb{C})^2-\{|X|+|Y|=0\}$ as follows:

$$f: \begin{pmatrix} X \\ Y \end{pmatrix} \rightarrow \begin{pmatrix} \zeta E & \eta \\ \tilde{\eta} & (1-\zeta)E \end{pmatrix} = \frac{1}{|X|+|Y|}\begin{pmatrix} |X|E & X\tilde{Y} \\ Y\tilde{X} & |Y|E \end{pmatrix} \quad (\text{where } \zeta \in \mathbb{C},\ \eta \in M_2(\mathbb{C}))$$

and this mapping is invariant by the group action $\mathbf{R}_g$: $\begin{pmatrix} X \\ Y \end{pmatrix} \rightarrow \begin{pmatrix} Xg \\ Yg \end{pmatrix}$, $g \in SL(2, \mathbb{C})$ moreover by the property of the above matrix ,i.e., $m^2=m$. the following relation holds.

$$\zeta^2+|\eta|=\zeta,$$
$$\therefore\ (1-2\zeta)^2+|2\eta|=1.$$

therefore we define an extended Hopf fiber bundle as follows:
Let $z_0=1-2\zeta$ and $Z=2\eta$ then $(z_0)^2+|Z|=1$.

Definition 4. (the <u>extended</u> Hopf fiber bundle)

We define the extended Hoph fiber bundle with structure group SL(2, $\mathbb{C}$) as follows:

$$S^7(\mathbb{C}) = \left\{ \begin{pmatrix} X \\ Y \end{pmatrix} \in M_2(\mathbb{C})^2 / |X| + |Y| = 1 \right\}$$

$$\pi = R_g \quad \Big| \quad SL(2, \mathbb{C}) \text{ (str. gr.)}$$

$$S^4(\mathbb{C}) = \{(z_0, Z) \in \mathbb{C} \times M_2(\mathbb{C}) / (z_0)^2 + |Z| = 1\}$$

(where the bundle space $S^7(\mathbb{C})$ is a complex 7-dimentional manifold and the base space $S^4(\mathbb{C})$ is a 4-dimentional manifold)

and the projection and structure group are

$$\pi : \begin{pmatrix} X \\ Y \end{pmatrix} \rightarrow (z_0, Z) = (|Y| - |X|, 2X\tilde{Y}),$$

$$G = SL(2, \mathbb{C}).$$

(when $z_0 \in \mathbb{R}$ and $Z \in \mathbb{R}^{1,3}$ for $S^4(\mathbb{C})$ then this base space is the anti-de Sitter space wich is a real 4-dimentional manifold.)

The base space $S^4(\mathbb{C})$ is coverd by two open sets which are correspond to the subset of C^4 by the following stereographic projection ϕ_i ($i = \pm 1$) as follows:

Two open sets are

$$S^4(\mathrm{Re}\, z_0 > -1) = S^4(\mathbb{C}) \cap \{\mathrm{Re}\, z_0 > -1\},$$
$$S^4(\mathrm{Re}\, z_0 < 1) = S^4(\mathbb{C}) \cap \{\mathrm{Re}\, z_0 < 1\}.$$

and by the stereographic projection ϕ_{-1} on $S^4(\mathrm{Re}\, z_0 > -1)$ and ϕ_1 on $S^4(\mathrm{Re}\, z_0 < 1)$,i.e.,

$$\phi_{-1} : (z_0, Z) \rightarrow u = Z(z_0 + 1)^{-1} = XY^{-1}$$

(see figure 2.)

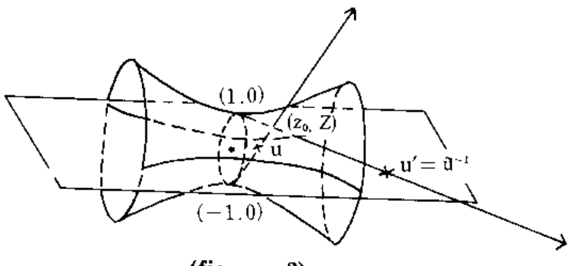
$$\phi_1 : (z_0, Z) \rightarrow u' = \tilde{Z}(1 - z_0)^{-1} = \tilde{X}^{-1}\tilde{Y}$$

this two parts are homeomorhic to the same part of the Euclidean space ,i.e.,

$$\mathbb{C}^4(\mathrm{Re}\, |u| > -1) = \{u \in M_2(\mathbb{C}) / \mathrm{Re}(1 + |u|) > 0\}$$

then the cross section on μ_{-1} on $\mathbb{C}^4(\mathrm{Re}|u| > -1)$ and μ_1 on $\mathbb{C}^4(\mathrm{Re}|u'| > -1)$ are as follows:

the anti-de Sitter space(real case)



(figure 2)

$$\mu_{-1}: u \rightarrow \frac{1}{(1+|u|)^{1/2}} \begin{pmatrix} u \\ \\ E \end{pmatrix} \in S^7(\mathbb{C})$$

$$u_1: u' \rightarrow \frac{1}{(1+|u|)^{1/2}} \begin{pmatrix} E \\ \\ \tilde{u}' \end{pmatrix} \in S^7(\mathbb{C})$$

Definition 5. (the connection on a principal fiber bundle)

Let $\mathbf{P}$ ($\mathbf{M}$, $\mathbf{G}$) be a principal fiber bundle over a manifold $\mathbf{M}$ with group $\mathbf{G}$. For each u $\in$ $\mathbf{P}$ Let $\mathbf{T}_p(\mathbf{P})$ be the tangent space of $\mathbf{P}$ at p and $\mathbf{G}_p$ the subspace of $\mathbf{T}_p(\mathbf{P})$ consisting of vectors tangent to the fiber through p. A connection Γ in $\mathbf{P}$ is an assignment of a subspace$\mathbf{Q}_p$ of $\mathbf{T}_p(\mathbf{P})$ to each p $\in$ $\mathbf{P}$ such that

(a) $\mathbf{T}_p(\mathbf{P})=\mathbf{G}_p+\mathbf{Q}_p$ (direct sum)

(b) $\mathbf{Q}_{pa}=(\mathbf{R}_a)_* \mathbf{Q}_p$ for every p $\in$ $\mathbf{P}$ and a $\in$ $\mathbf{G}$, where $\mathbf{R}_a$ is the transformation of $\mathbf{P}$ induced by a $\in$ $\mathbf{G}$, $\mathbf{R}_a p=pa$;

(c) $\mathbf{Q}_p$ depends differentiably on p.

We call $\mathbf{G}_p$ the vertical subspace and $\mathbf{Q}_p$ the horizontal subspace of $\mathbf{T}_p(\mathbf{P})$.

Proposition 6. (the canonical connection on the extended Hopf fiber bundle)

There is a canonical connection on the extended Hoph fiber bundle ,i.e., it's horizontal space is

$$\mathbf{Q}_p=\left\{ \begin{pmatrix} dX \\ \\ dY \end{pmatrix} \begin{matrix} \text{tangent} \\ \text{vector} \\ \text{at p} \end{matrix} \;\middle|\; \tilde{X}dX+\tilde{Y}dY=0 \text{ (matrix)} \right\}, \ {}^{\forall}p=\begin{pmatrix} X \\ \\ Y \end{pmatrix} \in \mathbf{P}.$$

Proof

Let P be the extended Hopf fiber bundle in the definition 4.

[the condition (c) in the definition 5]

The horizontal space

$$\mathbf{Q}_p=\left\{ \begin{pmatrix} dX \\ \\ dY \end{pmatrix} \begin{matrix} \text{tangent} \\ \text{vector} \\ \text{at p} \end{matrix} \;\middle|\; \tilde{X}dX+\tilde{Y}dY=0 \text{ (matrix)} \right\}, \ {}^{\forall}p=\begin{pmatrix} X \\ \\ Y \end{pmatrix} \in \mathbf{P}.$$

is complex 4-dimentional space and depends differentiably on p.

because

when $|Y| \neq 0$ then $dY=-\tilde{u}dX=-\tilde{u}(duY+udY)=-\tilde{u}duY-|u|dY \;\therefore dY=-(1+|u|)^{-1}\tilde{u}du\,Y$

$(\because u=XY^{-1} \in \mathbb{C}^4$ and $dX=duY+udY)$

when $|X| \neq 0$ then $dX=-u'dY=-u'(d\tilde{u}'X+\tilde{u}'dX)=-u'd\tilde{u}'X-|u|dX \;\therefore dX=-(1+|u|)^{-1}u'd\tilde{u}'X$

$(\because u'=\tilde{X}^{-1}\tilde{Y} \in \mathbb{C}^4$ and $dY=d\tilde{u}'X+\tilde{u}'dX)$

[the condition (a) in the definition 5]

For the element A $\in$ sl(2, $\mathbb{C}$), the relation $\frac{d}{dt}\begin{pmatrix} Xe^{tA} \\ \\ Ye^{tA} \end{pmatrix}_{t=0} = \begin{pmatrix} XA \\ \\ YA \end{pmatrix}$ holds.

Therefore the subspace of $\mathbf{T}_P(\mathbf{P})$

$$\mathbf{G}_P = \left\{ \begin{pmatrix} XA \\ \\ YA \end{pmatrix} \middle| \ A \in sl(2, \mathbb{C}) \right\}, \qquad {}^{\forall} p = \begin{pmatrix} X \\ \\ Y \end{pmatrix} \in P$$

is a set of consisting of vectors tangent to the fiber through p

when $A \neq 0$ then $dX = XA$, $dY = YA$ therefore $\tilde{X}dX + \tilde{Y}dY = (|X| + |Y|)A \neq 0$

this means that

$\mathbf{T}_P(\mathbf{P}) = \mathbf{G}_P + \mathbf{Q}_P$ (direct sum)

[the condition (b) in the definition 5]

The invariance for the action on fiber is as follows:

$$\mathbf{Q}_P = \left\{ \begin{pmatrix} dX_0 \\ \\ dY_0 \end{pmatrix} \begin{matrix} \text{tangent} \\ \text{vector} \\ \text{at pa} \end{matrix} \middle| \ \tilde{X}adX_0 + \tilde{Y}adY_0 = 0(\text{matrix}) \right\}, \ pa = \begin{pmatrix} Xa \\ \\ Ya \end{pmatrix} \in \mathbf{P}, \ {}^{\forall} a \in sl(2, \mathbb{C})$$

$$= \left\{ \begin{pmatrix} dXa \\ \\ dYa \end{pmatrix} \begin{matrix} \text{tangent} \\ \text{vector} \\ \text{at pa} \end{matrix} \middle| \ \tilde{X}dX + \tilde{Y}dY = 0(\text{matrix}) \right\}, \ {}^{\forall} p = \begin{pmatrix} X \\ \\ Y \end{pmatrix} \in \mathbf{P}$$

$$(\in \tilde{X}adX_0 + \tilde{Y}adY_0 = \tilde{a}(\tilde{X}dX_0a^{-1} + \tilde{Y}dY_0a^{-1})a)$$

$$= (\mathbf{R}_a)_* \, \mathbf{Q}_P$$

q.e.d.

The tangent vector $\begin{pmatrix} dX \\ \\ dY \end{pmatrix} \in \mathbf{T}_P(\mathbf{P})$ satisfy the following condition ,i.e.,

$$\mathrm{Re}\ (\tilde{X}dX + \tilde{Y}dY) = 0$$

and the connection from Θ on the principal fiber bundle satisfy the following horizntal condition ,i.e.,

$$\begin{pmatrix} dX \\ \\ dY \end{pmatrix} - \begin{pmatrix} X \\ \\ Y \end{pmatrix} \Theta_P, \ {}^{\forall} p = \begin{pmatrix} X \\ \\ Y \end{pmatrix} \in \mathbf{P}$$

is a horizontal vector.

therefore

$$(\tilde{X}dX + \tilde{Y}dY) - (|X| + |Y|)\Theta_P = 0$$

$$\therefore \quad \Theta_P\left(\begin{pmatrix} dX \\ \\ dY \end{pmatrix} \right) = \mathrm{Im}(\tilde{X}dX + \tilde{Y}dY)$$

Moreover the connection A(u) on the base is the pull back of Θ by the cross section μ_i $(i = \pm 1)$,i.e.,

$$A(u) = (\mu_{-1}{}^*\Theta)(u)$$

$$= \frac{1}{1 + |u|} \mathrm{Im}\ \tilde{u}du \qquad u \in \mathbb{C}^4(\mathrm{Re}\ |u| > -1)$$

$$A(u') = (\mu_1{}^*\Theta)(u')$$

$$= \frac{1}{1 + |u'|} \mathrm{Im}\ u'd\tilde{u}' \qquad u' \in \mathbb{C}^4(\mathrm{Re}\ |u'| > -1)$$

§ 3. The "Pseudo" - fiber bundle

Definition 7. (the pseudo-fiber bundle).

Let **M** be a manifold and **G** a Lie group. A pseudo fiber bundle **P** over **M** with group G is as follows:

(1) For the every point v of **P**, there is a neighborhood $\mathbf{P}_v$ and a projection π_v: $\mathbf{P}_v \to \mathbf{M}$ and that $\mathbf{P}_v(\mathbf{M}_v, \mathbf{G}_v, \pi_v)$, $\mathbf{M}_v = \pi_v(\mathbf{P}_v)$, $\mathbf{G}_v = \mathbf{G}$ is a principal fiber bundle

(2) **M** is covered by $\{\mathbf{M}_v / v \in \mathbf{P}\}$

(3) For any point $x \in \mathbf{M}_v \cap \mathbf{M}_w$ there exist $u \in \mathbf{P}_v \cap \mathbf{P}_w$ such that $\pi_v(u) = \pi_w(u) = x$, and the correspondence $x \to u$ is differentiable.

We call **P** the bundle space, **M** the base space, **G** the structure group.
and we call that **P** is covered by the principal fiber bundles $\mathbf{P}_v$

Propotition 8.
The condition (3) (in the above definition 7) is equal to the following condition (3)'
(3)' If $\mathbf{M}_v \cap \mathbf{M}_w \neq \Phi$ than there is a common local cross section $\sigma_{v,w}(x)$ on $\mathbf{M}_v \cap \mathbf{M}_w$.

proof
(3) $\Rightarrow$ (3)'

Let $\sigma_v(x)$, $\sigma_w(x)$ be each local cross section on $\mathbf{M}_v$, $\mathbf{M}_w$.

For any point $x \in \mathbf{M}_v \cap \mathbf{M}_w$, there exist $g_v(x) \in \mathbf{G}_v$, $g_w(x) \in \mathbf{G}_w$ such that $u(x) = \sigma_v(x)g_v(x) = \sigma_w(x)g_w(x)$. then $\sigma_{v,w}(x) = u(x)$ is a common local cross section on $\mathbf{M}_v \cap \mathbf{M}_w$

(3)' $\Rightarrow$ (3)

The common local cross section $u(x) = \sigma_{v,w}(x)$ on $\mathbf{M}_v \cap \mathbf{M}_w$ is also a local cross section in each principal fiber bundle $\mathbf{P}_v(\mathbf{M}_v, \mathbf{G}_v, \pi_v)$ and $\mathbf{P}_w(\mathbf{M}_w, \mathbf{G}_w, \pi_w)$. therefore $x = \pi_v(u) = \pi_w(u)$. q.e.d.

Theorem 9. (example of the pseudo-fiber bundle)
The complex 7-dimentional manifold

$$S^7(\mathbb{C})^E = \left\{ \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \in \mathbb{M}\mathbb{P}_2 / |X| + |Y| = |Z| \right\}$$

equipped with the structure of the pseudo-fiber bundle.

where space $\mathbb{M}\mathbb{P}_2$ is a projective space such that we identify two point of $(M_2(\mathbb{C}))^3 - \{|X| = |Y| = |Z| = 0\}$ by the following group action

$$\mathbf{R}_g: \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \\ \mathbf{Z} \end{pmatrix} \to \begin{pmatrix} \mathbf{Xg} \\ \mathbf{Yg} \\ \mathbf{Zg} \end{pmatrix}, \mathbf{g} \in SL(2, \mathbb{C})$$

proof
[the condition (1) in the definition 7]

The manifold $S^7(\mathbb{C})^E$ is covered by three principal fiber bundle as follows:

For any point $p=\begin{pmatrix} X_o \\ Y_o \\ Z_o \end{pmatrix} \in S^7(\mathbb{C})^E$, there are three cases (A) $|Z_o| \neq 0$, (B) $|Y_o| \neq 0$, (C) $|Y_o| \neq 0$

(A) When $|Z_o| \neq 0$

Let

$$S^7(u)=\left\{ \begin{pmatrix} X \\ Y \\ E \end{pmatrix} \in \mathrm{IM}\mathbb{P}_2 / |X|+|Y|=1 \right\},\ S^4(u)= \{(u_o, U) \in \mathbb{C}^5 / (u_o)^2+|U|=1\}$$

and

$$\pi_u: \begin{pmatrix} X \\ Y \\ E \end{pmatrix} \to (|Y|-|X|,\ 2X\tilde{Y})$$

then there is a neighborhood of p ,i.e., the principal fiber bundle

$$\begin{array}{ccc} S^7(u) & & \\ \pi_u \downarrow & & SL(2, \mathbb{C}) \\ S^4(u) & & \end{array}$$

(B) When $|X_o| \in 0$

Let

$$S^7(v)=\left\{ \begin{pmatrix} E \\ Y \\ Z \end{pmatrix} \in \mathrm{IM}\mathbb{P}_2 / 1+|Y|=|Z| \right\},\ S^4(v)= \{(v_o, V) \in \mathbb{C}^5 / (v_o)^2-|V|=1\}$$

and

$$\pi_V: \begin{pmatrix} E \\ Y \\ Z \end{pmatrix} \to (|Z|+|Y|,\ 2Y\tilde{Z})$$

there is a neighborhood of p ,i.e., the principal fiber bundle

$$\begin{array}{ccc} S^7(v) & & \\ \pi_V \downarrow & & SL(2, \mathbb{C}) \\ S^4(v) & & \end{array}$$

(C) When $|Y_o| \neq 0$

Let

$$S^7(w)=\left\{ \begin{pmatrix} X \\ E \\ Z \end{pmatrix} \in \mathrm{IM}\mathbb{P}_2 / |X|+1=|Z| \right\},\ S^4(w)= \{(w_o, W) \in \mathbb{C}^5 / (w_o)^2-|W|=1\}$$

and

$$\pi_W: \begin{pmatrix} X \\ E \\ Z \end{pmatrix} \to (-|X|-|Z|,\ 2Z\tilde{X})$$

there is a neighborhood of p ,i.e., the principal fiber bundle

$$\begin{array}{c} S^7(w) \\ \pi_W \Big\downarrow \quad SL(2, \mathbb{C}) \\ S^4(w) \end{array}$$

[the conditions (2), (3)' in the definition 7]

The base space $S^4(\mathbb{C})^E$ of the pseudo fiber bundle is as follows:
Let

$S^4(\mathbb{C})^E = S^4(u) \cup S^4(v) \cup S^4(w)$ (the disjoint union)

and we identify the two points between the following open sets

$$S^4(u) = S^4(\mathrm{Re}\ u_0 > -1) \cup S^4(\mathrm{Re}\ u_0 < 1)$$
$$S^4(v) = S^4(\mathrm{Re}\ v_0 > -1) \cup S^4(\mathrm{Re}\ v_0 < 1)$$
$$S^4(w) = S^4(\mathrm{Re}\ w_0 > -1) \cup S^4(\mathrm{Re}\ w_0 < 1)$$

according to the following cases (a), (b), (c).

(a) We identify $S^4(\mathrm{Re}\ u_0 < 1)$ and $S^4(\mathrm{Re}\ v_0 > 1)$ and the correspondence between u_0 and v_0 is as follows:

$$\boxed{\begin{array}{l} u_0:\ -\infty \cdots -1 \cdots 1 \\ v_0:\ \ -1 \cdots\ \ 1 \cdots \infty \end{array}}$$

(see figure 3)

[The cross section on S^4 (Re $U_0 < 1$)]

the stereographic projection $\phi_{u,1}$ on $S^4(\mathrm{Re}\ u_0 < 1)$ is

$$\phi_{u,1}: (u_0, U) \to u' = U(1-u_0)^{-1} = \tilde{X}^{-1}\tilde{Y} \in \mathbb{C}^4(\mathrm{Re}|u'| > -1)$$

and the cross section on $\mathbb{C}^4(\mathrm{Re}|u'| > -1)$ is

$$\mu_{u,1}: u' \to \begin{pmatrix} (1+|u'|)^{-1/2}E \\ \tilde{u}'(1+|u'|)^{-1/2} \\ E \end{pmatrix} \in S^7(u) \subset S^7(\mathbb{C})^E$$

[the cross section on S^4 (Re $v_0 > -1$)]

the stereographic projection $\phi_{v,-1}$ on $S^4(\mathrm{Re}\ v_0 > -1)$ is

$$\phi_{v,-1}: (v_0, V) \to v = V(v_0+1)^{-1} = YZ^{-1} \in \mathbb{C}^4(\mathrm{Re}|u| > -1)$$

and the cross section on $\mathbb{C}^4(\mathrm{Re}|v| > -1)$ is

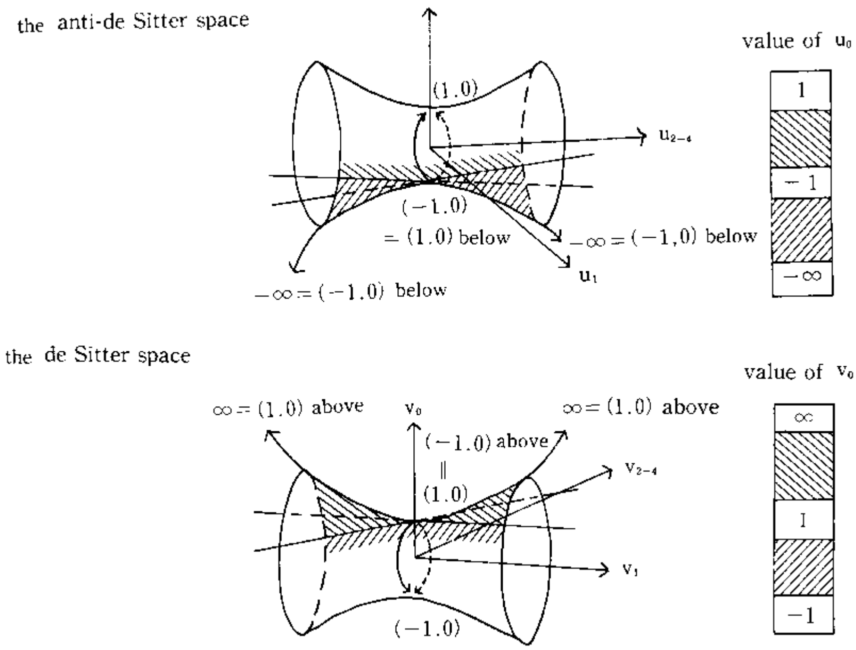
$$\mu_{v,-1}: v \to \begin{pmatrix} E \\ v(1-|u|)^{-1/2} \\ (1-|v|)^{-1/2}E \end{pmatrix} \in S^7(v) \subset S^7(\mathbb{C})^E$$

The coordinate transformation on $\mathbb{C}^4(\mathrm{Re}|u'|>-1)$ and $\mathbb{C}^4(\mathrm{Re}|v|>-1)$ is

$$u'=\tilde{v}(1-|v|)^{-1/2} \text{ (or } v=\tilde{u}'(1+|u'|)^{-1/2})$$

and above two cross sections are coincide by this coordinate transformation.

[the figure of the real case]



(figure 3)

(b) We identify $S^4(\mathrm{Re}\ v_0>-1)$ and $S^4(\mathrm{Re}\ w_0<1)$ and the correspondence between v_0 and w_0 is as follows:

v_0:	$-\infty$	$\cdots$	-1	$\cdots$	1
w_0:	-1	$\cdots$	1	$\cdots$	∞

[the cross section on S^4 (Re $v_0<1$)]

the stereographic projection $\phi_{v,1}$ on $S^4(\mathrm{Re}\ v_0<1)$ is

$$\phi_{v,1}: (v_0, V)\to v'=V(1-u_0)^{-1}=-\tilde{Y}^{-1}\tilde{Z} \in \mathbb{C}^4(\mathrm{Re}|v'|<1)$$

and the cross section on $\mathbb{C}^4(\mathrm{Re}|v'|<1)$ is

$$\mu_{v,1}: v'\to \begin{pmatrix} E \\ -i(1-|v'|)^{-1/2}E \\ -i\tilde{v}'(1-|v'|)^{-1/2} \end{pmatrix} \in S^7(v)\subset S^7(\mathbb{C})^E$$

the cross section on S^4 $(\mathrm{Re}\ w_0<1)$

the stereographic projection $\phi_{w,-1}$ on $S^4(\mathrm{Re}\ w_0>-1)$ is

$$\phi_{w,-1}: (w_0, W)\to w=W(w_0+1)^{-1}=ZX^{-1} \in \mathbb{C}^4(\mathrm{Re}|w|<1)$$

and the cross section on $\mathbb{C}^4(\mathrm{Re}|w|<1)$ is

$$\mu_{w,-1}: w\to \begin{pmatrix} i(1-|w|)^{-1/2}E \\ E \\ iw(1-|w|)^{-1/2} \end{pmatrix} \in S^7(w)\subset S^7(\mathbb{C})^E$$

The coordinate transformation on $\mathbb{C}^4(\mathrm{Re}|v'|<1)$ and $\mathbb{C}^4(\mathrm{Re}|w|<1)$ is

$$v' = iw(1-|w|)^{-1/2}(\text{or } w=iv'(1-|v'|)^{-1/2})$$

and above two cross sections are coincide by this coordinate transformation.

(c) We identify $S^4(\mathrm{Re}\ w_0<1)$ and $S^4(\mathrm{Re}\ u_0>-1)$ and the correspondence between w_0 and u_0 is as follows:

w_0:	$-\infty$	$\cdots -1$	$\cdots$	1	
u_0:	-1	$\cdots$	1	$\cdots\infty$	

the stereographic projection $\phi_{w,1}$ on $S^4(\mathrm{Re}\ w_0<1)$ is

$$\phi_{w,1}: (w_0, W)\to w'=W(1-w_0)^{-1}=\tilde{Z}^{-1}\tilde{X} \in \mathbb{C}^4(\mathrm{Re}|w'|<1)$$

and the cross section on $\mathbb{C}^4(\mathrm{Re}|w'|<1)$ is

$$\mu_{w,1}: w'\to \begin{pmatrix} \tilde{w}(1-|w'|)^{-1/2} \\ E \\ (1-|w'|)^{-1/2}E \end{pmatrix} \in S^7(w)\subset S^7(\mathbb{C})^E$$

〔the cross section on S^4 (Re $u_0 > -1$)〕

the stereographic projection $\phi_{w,-1}$ on S^4(Re $u_0 > -1$) is

$$\phi_{u,-1}: (u_0, U) \to u = U(u_0+1)^{-1} = XY^{-1} \in \mathbb{C}^4(\mathrm{Re}|u| > -1)$$

and the cross section on $\mathbb{C}^4(\mathrm{Re}|u| > -1)$ is

$$\mu_{u,-1}: u \to \begin{pmatrix} u(1+|u|)^{-1/2} \\ (1+|u|)^{-1/2}E \\ E \end{pmatrix} \in S^7(u) \subset S_7(\mathbb{C})^E$$

The coordinate transformation on $\mathbb{C}^4(\mathrm{Re}|u| > -1)$ and $\mathbb{C}^4(\mathrm{Re}|w'| < 1)$ is

$$w' = \hat{u}(1+|u|)^{-1/2} \text{ (or } u = \hat{w}'(1-|w'|)^{-1/2})$$

and above two cross sections are coincide by this coordinate transformation.

The following proposition holds.

Proposition 10.

The base space $S^4(\mathbb{C})^E$ is divided into 3-parts as follows:

$S^4(\mathbb{C})^E = S^4(u,-1,1) \cup S^4(v,-1,1) \cup S^4(w,-1,1)$

and $S^4(u,-1,1) \cap S^4(v,-1,1) = S^4(v,-1,1) \cap S^4(w,-1,1) = S^4(w,-1,1) \cap S^4(u,-1,1) = \phi$

where

$S^4(u,-1,1) = S^4(u) \cap \{-1 < \mathrm{Re}\, u_0 \leqq 1\}$

$S^4(v,-1,1) = S^4(v) \cap \{-1 < \mathrm{Re}\, v_0 \leqq 1\}$

$S^4(w,-1,1) = S^4(w) \cap \{-1 < \mathrm{Re}\, w_0 \leqq 1\}$

§ 4. The Connection on the Pseudo-Fiber Bundle

Definition 11. (the connection on the pseudo fiber bundle)

Let **P** (**M**, **G**) be a pseudo fiber bundle over a manifold **M** with group **G**. A connection Γ on **P** is an assignment of a subspace $\mathbf{Q}_P$ of $\mathbf{T}_P(\mathbf{P})$ to each p ∈ **P** such that on the principal fiber bundle which is a neighborhood of p there exist a connection and it's horizontal subspace is $\mathbf{Q}_P$.

We call $\mathbf{Q}_P$ the horizontal subspace, Γ the connection.

Theorem 12.

There is a canonical connection on the pseudo-fiber bundle **P** (in theorem 8) and it's horizontal subspace is

$$\mathbf{Q}_P=\left\{ \begin{pmatrix} dX \\ dY \\ dZ \end{pmatrix} \begin{matrix} \text{tangent} \\ \text{vector} \\ \text{at } p \end{matrix} \middle| \ \tilde{X}dX+\tilde{Y}dY-\tilde{Z}dZ=0(\text{matrix}) \right\},\ {}^{\forall} p=\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \in \mathbf{P}$$

Proof

For any point $p=\begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \end{pmatrix}$, there are three cases (A) $|Z_0|\neq 0$, (B) $|X_0|\neq 0$, (C) $|Y_0|\neq 0$

For the case (A), ie., $|Z_0|\neq 0$

There is a principal fiber bundle $S^7(u)=\left\{ \begin{pmatrix} X \\ Y \\ E \end{pmatrix} \in \mathbb{M}\mathbb{P}_2 / |X|+|Y|=1 \right\}$

[the condition (c) in the definition 5]

The horizontal space

$$\mathbf{Q}_u=\left\{ \begin{pmatrix} dX_0 \\ dY_0 \\ 0 \end{pmatrix} \begin{matrix} \text{tangent} \\ \text{vector} \\ \text{at } u \end{matrix} \middle| \ \widetilde{XZ^{-1}}dX_0+\widetilde{YZ^{-1}}dY_0=0 \right\},\ {}^{\forall} u=\begin{pmatrix} XZ^{-1} \\ YZ^{-1} \\ E \end{pmatrix} \in S^7(u)$$

is complex 4-dimentional subspace and depends differentiably on u.

because $\quad \widetilde{XZ^{-1}}d(XZ^{-1})+\widetilde{YZ^{-1}}d(YZ^{-1})=\tilde{Z}(\tilde{X}dX+\tilde{Y}dY-\tilde{Z}dZ)Z=0$

$$\therefore d(YZ^{-1})=\tilde{Y}^{-1}\tilde{X}d(XZ^{-1}) \text{ or } d(XZ^{-1})=\tilde{X}^{-1}\tilde{Y}d(YZ^{-1})$$

[the condition (a) in the definition 5]

For the element $A\in sl(2,\mathbb{C})$, the relation $\dfrac{d}{dt}\begin{pmatrix} X \\ Y \\ e^{-tA}Z \end{pmatrix}_{t=0}=\begin{pmatrix} 0 \\ 0 \\ -AZ \end{pmatrix}$ holds.

Therefore the subspace of $\mathbf{T}_u(S^7(u))$

$$\mathbf{G}_u=\left\{ \begin{pmatrix} 0 \\ 0 \\ -AZ \end{pmatrix} \middle| \ A\in sl(2,\mathbb{C}) \right\}\ {}^{\forall} u=\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \in S^7(u)$$

is a set of consisting of vectors tangent to the fiber through u.

When $A\neq 0$ then $dX=0$, $dY=0$, $dZ=AZ$ therefore $\tilde{X}dX+\tilde{Y}dY-\tilde{Z}dZ=\tilde{X}\cdot 0+\tilde{Y}\cdot 0-\tilde{Z}\cdot(-AZ)\neq 0(\text{matrix})$

this means that

$\mathbf{T}_u(S^7(u))=\mathbf{G}_u+\mathbf{Q}_u$ (direct sum).

[the condition (b) in the definition 5]

$$\mathbf{Q}_{ua}=\left\{ \begin{pmatrix} dX \\ dY \\ dZ_0 \end{pmatrix} \begin{matrix} \text{tangent} \\ \text{vector} \\ \text{at } ua \end{matrix} \middle| \ \tilde{X}dX+\tilde{Y}dY-a^{-1}ZdZ=0 \right\},\ ua-\begin{pmatrix} X \\ Y \\ a^{-1}Z \end{pmatrix} \in S^7(u),\ {}^{\forall} a \in SL(2,\mathbb{C})$$

$$= \left\{ \left. \begin{pmatrix} dX \\ dY \\ a^{-1}dZ \end{pmatrix} \begin{matrix} \text{tangent} \\ \text{vector} \\ \text{at ua} \end{matrix} \right| \tilde{X}dX+\tilde{Y}dY-\tilde{Z}dZ=0 \right\} \ {}^{\forall} u= \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \in S^7(u)$$

$(\because \tilde{X}dX+\tilde{Y}dY-(a^{-1}Z)dZ_0=\tilde{X}dX+\tilde{Y}dY-\tilde{Z}adZ_0)$

$=(\mathbf{R}_a)_* \mathbf{Q}_u$

The same statements are true for the cases (B) and (C).

q.e.d.

The tangent vector $\begin{pmatrix} dX \\ dY \\ dZ \end{pmatrix} \in \mathbf{T}_p(\mathbf{P})$ satisfy the following condition ,i.e.,

$$\mathrm{Re}(\tilde{X}dX+\tilde{Y}dY) = \mathrm{Re}(\tilde{Z}dZ) \qquad \text{because} \quad |X|+|Y|=|Z|$$

and the connection form Θ on each principal fiber bundle satisfy the following horizontal condition ,i.e.,

$$\begin{pmatrix} dX \\ dY \\ dz \end{pmatrix} - \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \Theta(p), \ {}^{\forall} p= \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \in \mathbf{P}, \ (\ |X| \neq 0 \text{ or} |Z| \neq 0 \text{ or} |Y| \neq 0)$$

is a horizontal vector.

Hence we restrict the connection from Θ to the three cases (A), (B) and (C).

For the case (A) ,i.e., $|Z| \neq 0$,

$$(\tilde{X}dX+\tilde{Y}dY-\tilde{Z}dZ)-(\tilde{X}\cdot 0+\tilde{Y}\cdot 0-\tilde{Z}\cdot(-\Theta(u)Z))=0$$

$$\therefore \Theta(u)=\tilde{Z}^{-1}(\tilde{X}dX+\tilde{Y}dY-\tilde{Z}dZ)Z^{-1} = \tilde{Z}^{-1}\mathrm{Im}(\tilde{X}dX+\tilde{Y}dY-\tilde{Z}dZ)Z^{-1}=\mathrm{Im}(\tilde{X}dX+\tilde{Y}dY)$$

moreover pull back of $\Theta(u)$ by the cross section $\mu_{u,1}$ and $\mu_{u,-1}$ is as follows.

$A(u)=\frac{1}{1+\lvert u\rvert} \mathrm{Im}(\tilde{u}du) \qquad \text{on } \mathbb{C}^4(\mathrm{Re}\lvert u\rvert>-1)$
$A(w')=\frac{1}{1+\lvert u'\rvert} \mathrm{Im}(u'\, d\tilde{u}') \qquad \text{on } \mathbb{C}^4(\mathrm{Re}\lvert u'\rvert>-1)$

For the case (B) ,i.e., $|Y| \neq 0$,

$$(\tilde{X}dX+\tilde{Y}dY-\tilde{Z}dZ)-(\tilde{X}\cdot(-\Theta(\tilde{w})X)+Y\cdot 0-Z\cdot 0)=0$$

$$\therefore \Theta(v)=\tilde{X}^{-1}(\tilde{Z}dZ-\tilde{X}dX-\tilde{Y}1dY)X^{-1} = \tilde{X}^{-1}\mathrm{Im}(\tilde{Z}dZ+\tilde{X}dX-\tilde{Y}dY)X^{-1} = \mathrm{Im}(\tilde{Z}dZ-\tilde{Y}dY)$$

Moreover pull back of $\Theta(v)$ by the cross section $\mu_{v,1}$ and $\mu_{v,-1}$ is as follows.

$C(v)=\frac{1}{\lvert v\rvert-1} \mathrm{Im}(\tilde{v}dv) \qquad \text{on } \mathbb{C}^4(\mathrm{Re}\lvert v\rvert<1)$
$C(v')=\frac{1}{\lvert v'\rvert-1} \mathrm{Im}(v'\, d\tilde{v}') \qquad \text{on } \mathbb{C}^4(\mathrm{Re}\lvert v'\rvert<1)$

For the case (C) ,i.e., $|X| \neq 0$,

$$(\tilde{X}dX+\tilde{Y}dY-\tilde{Z}dZ)-(\tilde{X}\cdot 0+\tilde{Y}\cdot(-\Theta(v)Y)-\tilde{Z}\cdot 0)=0$$

$$\therefore \quad \Theta(w) = \tilde{Y}^{-1}(\tilde{Z}dZ - \tilde{X}dX - \tilde{Y}dY)Y^{-1} = \tilde{Y}^{-1}\mathrm{Im}(\tilde{Z}dZ - \tilde{X}dX - \tilde{Y}dY)Y^{-1} = \mathrm{Im}(\tilde{Z}dZ - \tilde{X}dX)$$

Moreover pull back of $\Theta(w)$ by the cross section $\mu_{w,1}$ and $\mu_{w,-1}$ is as follows.

$$\boxed{\begin{array}{ll} B(w) = \dfrac{1}{|w|-1}\,\mathrm{Im}(\tilde{w}dw) & \text{on } \mathbb{C}^4(\mathrm{Re}|v|<1) \\ B(w') = \dfrac{1}{|w'|-1}\,\mathrm{Im}(w'd\tilde{w}') & \text{on } \mathbb{C}^4(\mathrm{Re}|v'|<1) \end{array}}$$

The connection on the base of this pseudo fiber bundle is obtained by combining above six connection smoothly.

References

M.F.Atiyha N.J.Hitchin: CONSTRUCTION OF INSTANTONS, Phys. Lett. 65A-3 (1978) 185-187.
S.Helgason Differential: Geometry, Lie Groups, and Symmetric Spaces Academic Press, Inc. (1978)
Kobayashi and Nomizu: Foundations of Differential Geometry I Interscience Publishers(1969)