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The Transformation Group on an Extended Hopf Fiber Bundle and Its Associated Bundle*

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Abstract

We discuss the extended Hopf fiber bundle defined in a previous paper[3)] and its associated frame bundle.

In §2, we add the infinities of the base space in a 6-dimentional homogeneous coordinate to expand the above fiber bundle.

In §3, we see that under this situation, transfomation group $SL(4, \mathbb{C})$ on this fiber bundle acts on the base space as the group $SO(2,4, \mathbb{C})$ and we also find that a canonical connection (i.e., instanton) on the extended Hopf fiber bundle corresponds to the Levi-Civita connection on the associated frame bundle.

In §4, at last, we also construct a new structure of extended Hopf fiber bundle whose fiber preserves the Levi-Civita connection in the transformation group $SL(4, \mathbb{C})$ and then, make a cross section and a connection on this fiber bundle by the use of the decomposition of $SL(4, \mathbb{C})$ which is similar to the Iwasawa decomposition.

§1. Introduction

Let

$$M_2(\mathbb{C}) = \left\{ u = \begin{pmatrix} u_1+u_2 & u_3+iu_4 \\ u_3-iu_4 & u_1-u_2 \end{pmatrix} \middle/ u_1, u_2, u_3, u_4 \in \mathbb{C} \right\}.$$

$$\tilde{u} = \begin{pmatrix} u_1-u_2 & -u_3-iu_4 \\ -u_3+iu_4 & u_1+u_2 \end{pmatrix} \text{ a cofactor matrix of } u,$$

and $|u| = (u_1)^2-(u_2)^2-(u_3)^2-(u_4)^2$ a determinant of u.

We define the extended Hoph fiber bundle[3)] with structure group $SL(2, \mathbb{C})$ as follows:

$$S^7(\mathbb{C}) = \left\{ \begin{pmatrix} X \\ Y \end{pmatrix} \in M_2(\mathbb{C})^2 / |X|+|Y| = 1 \right\},$$

$$\pi \downarrow \quad R_g (g \in SL(2, \mathbb{C}))$$

$$S^4(\mathbb{C}) = \{(u_0, U) \in \mathbb{C} \times M_2(\mathbb{C}) / (u_0)^2 + |U| = 1\}.$$

The structure group is $SL(2, \mathbb{C})$ and the action R_g is

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$$R_g: \begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} Xg \\ Yg \end{pmatrix}, \ g \in SL(2, \mathbb{C})$$

and the projection π: $S^7(\mathbb{C}) \to S^4(\mathbb{C})$ is as follows:

We consider the mapping f: $S^7(\mathbb{C}) \to \{m \in M(4, \mathbb{C}) / {}^t\tilde{m} = m,\ m^2 = m,\ \mathrm{Tr}\, m = 1\}$,

$$f: \begin{pmatrix} X \\ Y \end{pmatrix} \to m = \begin{pmatrix} \zeta & \eta \\ \tilde{\eta} & 1-\zeta \end{pmatrix} = \frac{1}{|X|+|Y|} \begin{pmatrix} X\tilde{X} & X\tilde{Y} \\ Y\tilde{X} & Y\tilde{Y} \end{pmatrix} \quad (\text{where } \zeta \in \mathbb{C}E,\ \eta \in M_2(\mathbb{C})),$$

this mapping is invariant under the group action R_g: $\begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} Xg \\ Yg \end{pmatrix}$, $g \in SL(2, \mathbb{C})$ and surjective, moreover the following relation holds by the property of the above matrix (i.e., $m^2 = m$),

$$\zeta^2 + |\eta| = \zeta,$$

$$\therefore \quad (1-2\zeta)^2 + |2\eta| = 1 .$$

Therefore we can define the projection π, i.e.,

$$\pi: \begin{pmatrix} X \\ Y \end{pmatrix} \to (u_0, U) = (1-2\zeta, 2\eta) = \left(\frac{|Y|-|X|}{|Y|+|X|},\ \frac{2X\tilde{Y}}{|Y|+|X|} \right).$$

The bundle space $S^7(\mathbb{C})$ and the base space $S^4(\mathbb{C})$ is covered by two open sets which are correspond to the subset of $\mathbb{C}^4$ by the stereographic projection $\phi_i\ (i = \pm 1)$, i.e.,

Two open sets are

$$M_2(\mathbb{C})^2_{-1} = \left\{ \begin{pmatrix} X \\ Y \end{pmatrix} \in S^7(\mathbb{C}) / |Y| \neq 0 \right\} \xrightarrow{\pi} S^4(u_0 \neq -1) = S^4(\mathbb{C}) \cap \{u_0 \neq -1\},$$

and

$$M_2(\mathbb{C})^2_{1} = \left\{ \begin{pmatrix} X \\ Y \end{pmatrix} \in S^7(\mathbb{C}) / |X| \neq 0 \right\} \xrightarrow{\pi} S^4(u_0 \neq 1) = S^4(\mathbb{C}) \cap \{u_0 \neq 1\},$$

And by the stereographic projection ϕ_{-1} on $S^4(u_0 \neq -1)$ and ϕ_1 on $S^4(u_0 \neq 1)$, i.e.,

$$\phi_{-1}: (u_0, U) \to u = U(u_0+1)^{-1} = XY^{-1} \in M_2(\mathbb{C}), \qquad \text{(see figure 1.)}$$

$$\phi_1: (u_0, U) \to u' = U(1-u_0)^{-1} = \tilde{X}^{-1}\tilde{Y} (= \tilde{u}^{-1}) \in M_2(\mathbb{C}).$$

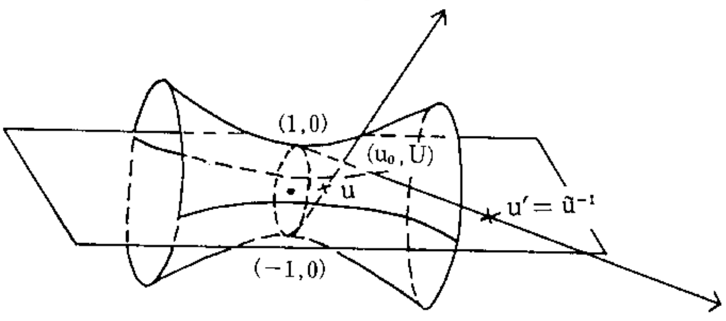
This two parts are homeomorphic to the same part of $\mathbb{C}^4$, i.e.,

$$\mathbb{C}^4(|u| \neq -1) = \{u \in M_2(\mathbb{C}) / 1 + |u| \neq 0\},$$

then the cross section on μ_{-1} on $\mathbb{C}^4(|u| \neq -1)$ and μ_1 on $\mathbb{C}^4(|u'| \neq -1)$ are as follows:

$$\mu_{-1}: u \to \frac{1}{(1+|u|)^{1/2}} \begin{pmatrix} u \\ E \end{pmatrix} \in M_2(\mathbb{C})^2_{-1},$$

the anti-de Sitter space(real case)



(figure 1.)

$$\mu_1 : u' \to \frac{1}{(1+|u'|)^{1/2}} \begin{pmatrix} E \\ \tilde{u}' \end{pmatrix} \in M_2(\mathbb{C})^2{}_1 .$$

There is a canonical connection on the extended Hoph fiber bundle by the horizontal condition ,i.e.,

$$< \begin{pmatrix} X \\ Y \end{pmatrix}, \begin{pmatrix} dX \\ dY \end{pmatrix} > = \tilde{X}dX + \tilde{Y}dY = 0(\text{matrix}) \quad \text{for} \begin{pmatrix} dX \\ dY \end{pmatrix} \in T_p(S^7(\mathbb{C})) \text{ at } \forall p = \begin{pmatrix} X \\ Y \end{pmatrix} \in P,$$

and the connection form Θ on the principal fiber bundle is

$$\Theta_p\left(\begin{pmatrix} dX \\ dY \end{pmatrix} \right) = \tilde{X}dX + \tilde{Y}dY \qquad \text{on } T_p(S^7(\mathbb{C})).$$

Moreover

The connection $A(u)$ on the base space is the pull back of Θ by the cross section μ_i $(i = \pm 1)$,i.e.,

$$A(u) = (\mu_{-1}{}^*\Theta)(u)$$

$$= \frac{1}{1+|u|} \operatorname{Im} \tilde{u}du \qquad u \in \mathbb{C}^4(|u| \neq -1),$$

$$A(u') = (\mu_1{}^*\Theta)(u')$$

$$= \frac{1}{1+|u'|} \operatorname{Im} u'd\tilde{u}' \qquad u' \in \mathbb{C}^4(|u'| \neq -1).$$

§ 2. An expansion of the base space and the cross section on it

We define a transformation on $M_2(\mathbb{C})^2$ of $SL(4, \mathbb{C})$ as follows:
Let $a, b, c, d, X, Y \in M_2(\mathbb{C})$ and

$$SL(4, \mathbb{C}) \ni g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} aX+bY \\ cX+dY \end{pmatrix}.$$

Theorem 1.

$M_2(\mathbb{C})^2{}_0$ is a invariant subspace of the transformation group $SL(4, \mathbb{C})$ and has the structure of the fiber bundle with structure group $GL(2, \mathbb{C})$ as follows:
the fiber space is

$$M_2(\mathbb{C})^2{}_0 = \left\{ \begin{pmatrix} X \\ Y \end{pmatrix} \in M_2(\mathbb{C})^2 \right\} \setminus \{|X| = |Y| = 0 \text{ and } X\tilde{Y} = 0\},$$

the base space is

$$\tilde{S}^4(\mathbb{C}) = \left\{ \begin{bmatrix} u_0 \\ U \\ u_5 \end{bmatrix} \in \mathbb{C}P_5 / (u_0)^2 + |U| = (u_5)^2, u_0, u_5 \in \mathbb{C} \text{ and } U = \begin{pmatrix} u_1+u_2 & u_3+iu_4 \\ u_3-iu_4 & u_1-u_2 \end{pmatrix} \in M_2(\mathbb{C}) \right\}$$

and the projection π is

$$\pi : \begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{bmatrix} u_0 \\ U \\ u_5 \end{bmatrix} = \begin{bmatrix} |Y| - |X| \\ 2X\tilde{Y} \\ |Y| + |X| \end{bmatrix}.$$

Proof

The subspace $M_2(\mathbb{C})^2{}_0$ is invariance under the action of $SL(4, \mathbb{C})$,because the following equations (1), (2),(3) hold.

(1) $|aX+bY| = (aX+bY)(\tilde{X}\tilde{a} + \tilde{Y}\tilde{b}) = |X||a| + aX\tilde{Y}\tilde{b} + bY\tilde{X}\tilde{a} + |Y||b| = 0,$

(2) $|cX+dY| = (cX+dY)(\tilde{X}\tilde{c} + \tilde{Y}\tilde{d}) = |X||c| + cX\tilde{Y}\tilde{d} + dY\tilde{X}\tilde{c} + |Y||d| = 0,$

(3) $(aX+bY)\widetilde{(cX+dY)} = (aX+bY)(\tilde{X}\tilde{c} + \tilde{Y}\tilde{d}) = |X|a\tilde{c} + aX\tilde{Y}\tilde{d} + bY\tilde{X}\tilde{c} + |Y|b\tilde{d} = 0.$

We exchange the coordinate of the base space $\tilde{S}^4(\mathbb{C})$ as follows:

$2u_{5-0}=u_5-u_0,\ 2u_{1-2}=u_1-u_2,\ 2u_{3-14}=u_3-iu_4,$

$2u_{5+0}=u_5+u_0,\ 2u_{1+2}=u_1+u_2,\ 2u_{3+14}=u_3+iu_4,$

then the base space $\tilde{S}^4(\mathbb{C})$ is represented as

$$|U| = 4u_{5+0}u_{5-0}\ ,\ U = 2\begin{pmatrix} u_{1+2} & u_{3+14} \\ u_{3-14} & u_{1-2} \end{pmatrix},$$

and the projection is

$$\begin{pmatrix} X \\ Y \end{pmatrix} \rightarrow \begin{bmatrix} u_{5-0} \\ U \\ u_{5+0} \end{bmatrix} = \begin{bmatrix} |X| \\ 2XY \\ |Y| \end{bmatrix}$$

We devide the base space $\tilde{S}^4(\mathbb{C})$ into 6-parts by the use of this coordinate ,i.e.,

$\tilde{S}^4(\mathbb{C}) = S^4(u_{5+0}) \cup S^4(u_{5-0}) \cup S^4(u_{1+2}) \cup S^4(u_{1-2}) \cup S^4(u_{3+14}) \cup S^4(u_{3-14})$,

(a) $S^4(u_{5+0})=\tilde{S}^4(\mathbb{C}) \cap \{u_{5+0}\neq 0\}$, (b) $\tilde{S}^4(u_{5-0})=\tilde{S}^4(\mathbb{C}) \cap \{u_{5-0}\neq 0\}$,

(c) $S^4(u_{1+2})=\tilde{S}^4(\mathbb{C}) \cap \{u_{1+2}\neq 0\}$, (d) $\tilde{S}^4(u_{1-2})=\tilde{S}^4(\mathbb{C}) \cap \{u_{1-2}\neq 0\}$,

(e) $S^4(u_{3+14})=\tilde{S}^4(\mathbb{C}) \cap \{u_{3+14}\neq 0\}$, (f) $\tilde{S}^4(u_{3-14})=\tilde{S}^4(\mathbb{C}) \cap \{u_{3-14}\neq 0\}$.

The cross sections on the above space (a)～(f) are as follows:

(a) The cross section on $S^4(u_{5+0})$ is

$$\begin{bmatrix} u_{5-0} \\ U \\ u_{5+0} \end{bmatrix} \rightarrow (u_{5+0})^{-1/2}\begin{pmatrix} U/2 \\ u_{5+0}E \end{pmatrix}.$$

(b) The cross section on $S^4(u_{5-0})$ is

$$\begin{bmatrix} u_{5-0} \\ U \\ u_{5+0} \end{bmatrix} \rightarrow (u_{5-0})^{-1/2}\begin{pmatrix} u_{5-0}E \\ U/2 \end{pmatrix}.$$

(c) The cross section on $S^4(u_{1+2})$ is

$$\begin{bmatrix} u_{5-0} \\ U \\ u_{5+0} \end{bmatrix} \rightarrow (u_{1+2})^{-1/2}\begin{pmatrix} \begin{pmatrix} u_{1+2} & 0 \\ u_{3-14} & u_{5-0} \end{pmatrix} \\ \begin{pmatrix} u_{5+0} & -u_{3+14} \\ 0 & u_{1+2} \end{pmatrix} \end{pmatrix},$$

where $u_{1-2}=(u_{1+2})^{-1}(u_{3-14}\ u_{3+14}+u_{5-0}\ u_{5+0})$.

(d) The cross section on and $S^4(u_{1-2})$ is

$$\begin{bmatrix} u_{5-0} \\ U \\ u_{5+0} \end{bmatrix} \rightarrow (u_{1-2})^{-1/2}\begin{pmatrix} \begin{pmatrix} u_{5-0} & u_{3+14} \\ 0 & u_{1-2} \end{pmatrix} \\ \begin{pmatrix} u_{1-2} & 0 \\ -u_{3-14} & u_{5+0} \end{pmatrix} \end{pmatrix},$$

where $u_{1+2}=(u_{1-2})^{-1}(u_{3-14}\ u_{3+14}+u_{5-0}\ u_{5+0})$.

(e) The cross section on $S^4(u_{3+14})$ is

$$\begin{bmatrix} u_{5-0} \\ U \\ u_{5+0} \end{bmatrix} \rightarrow (u_{3+14})^{-1/2} \begin{pmatrix} \begin{pmatrix} 0 & u_{3+14} \\ -u_{5-0} & u_{1-2} \end{pmatrix} \\ \begin{pmatrix} u_{3+14} & 0 \\ -u_{1+2} & u_{5+0} \end{pmatrix} \end{pmatrix},$$

where $u_{3-14} = (u_{3+14})^{-1}(u_{1+2}\, u_{1-2} - u_{5-0}\, u_{5+0})$.

(f) The cross section on $S^4(u_{3-14})$ is

$$\begin{bmatrix} u_{5-0} \\ U \\ u_{5+0} \end{bmatrix} \rightarrow (u_{3-14})^{-1/2} \begin{pmatrix} \begin{pmatrix} u_{1+2} & -u_{5-0} \\ u_{3-14} & 0 \end{pmatrix} \\ \begin{pmatrix} u_{5+0} & -u_{1-2} \\ 0 & u_{3-14} \end{pmatrix} \end{pmatrix},$$

where $u_{3+14} = (u_{3-14})^{-1}(u_{1+2}\, u_{1-2} - u_{5-0}\, u_{5+0})$.

The structure group is $GL(2, \mathbb{C})$, because

(a) When $u_{5+0} \neq 0(|Y| \neq 0)$

Let's for any point $\begin{pmatrix} X' \\ Y' \end{pmatrix} \in \tilde{S}^7(\mathbb{C})$,

$|X| = |X'|$, $X\hat{Y} = X'\hat{Y}'$, $|Y| = |Y'|$.

Then there exist $g = Y^{-1}Y' \in SL(2, \mathbb{C})$,

and

$Yg = Y'$,

$X = X'\tilde{g} = X'g^{-1} \quad \therefore \quad Xg = X'$.

(b) When $u_{5-0} \neq 0(|X| \neq 0)$, the same manner as $u_{5+0} \neq 0$ is done.

(c) When $u_{1+2} \neq 0(U_{1,1} \neq 0)$

Let's for any point $\begin{pmatrix} X' \\ Y' \end{pmatrix} \in \tilde{M}_2(\mathbb{C})^2{}_0$,

$|X| = |X'|$, $X\hat{Y} = X'\hat{Y}'$ and $|Y| = |Y'|$,

, i.e.,

(by $|X| = |X'|$)

$u_{5-0} = x_{1+2}\, x_{1-2} - x_{3+14}\, x_{3-14} \cdots (1)$.

(by $2X\hat{Y} = 2X'\hat{Y}'$)

$$2\begin{pmatrix} u_{1+2} & u_{3+14} \\ u_{3-14} & u_{1-2} \end{pmatrix} = 2\begin{pmatrix} x_{1+2} & x_{3+14} \\ x_{3-14} & x_{1-2} \end{pmatrix} \begin{pmatrix} y_{1-2} & -y_{3+14} \\ -y_{3-14} & y_{1+2} \end{pmatrix},$$

$\therefore$

$u_{1+2} = x_{1+2}\, y_{1-2} - x_{3+14}\, y_{3-14} \cdots\cdots\cdots (2)$,

$u_{3+14} = -x_{1+2}\, y_{3+14} + x_{3+14}\, y_{1+2} \cdots\cdots (3)$,

$u_{3-14} = x_{3-14}\, y_{1-2} - x_{1-2}\, y_{3-14} \cdots\cdots\cdots (4)$,

$u_{1-2} = -x_{3-14}\, y_{3+14} + x_{1-2}\, y_{1+2} \cdots\cdots (5)$.

(by $|Y| = |Y'|$)

$u_{5+0} = y_{1+2}\, y_{1-2} - y_{3+14}\, y_{3-14} \cdots (6)$.

Then there exist $g = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix}$ such that g_1, g_2, g_3, g_4 is satisfy the following (7), (12)

(by $Xg = X'$)

$$(u_{1+2})^{-1/2} \begin{pmatrix} u_{1+2} & 0 \\ u_{3-14} & u_{5-0} \end{pmatrix} \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} = \begin{pmatrix} x_{1+2} & x_{3+14} \\ x_{3-14} & x_{1-2} \end{pmatrix},$$

$\therefore (u_{1+2})^{1/2} g_1 = x_{1+2},\ (u_{1+2})^{1/2} g_2 = x_{3+14} \cdots\cdots (7),$

$u_{3-14} g_1 + u_{5-0} g_3 = (u_{1+2})^{1/2} x_{3-14} \cdots\cdots\cdots\cdots (8),$

$u_{3-14} g_2 + u_{5-0} g_4 = (u_{1+2})^{1/2} X_{1-2} \cdots\cdots\cdots\cdots (9).$

(by $Yg = Y'$)

$$(u_{1+2})^{-1/2} \begin{pmatrix} u_{5+0} & -u_{3+14} \\ 0 & u_{1+2} \end{pmatrix} \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} = \begin{pmatrix} y_{1+2} & y_{3+14} \\ y_{3-14} & y_{1-2} \end{pmatrix},$$

$\therefore u_{5+0} g_1 - u_{3+14} g_3 = (u_{1+2})^{1/2} y_{1+2} \cdots\cdots\cdots\cdots (10),$

$u_{5+0} g_2 - u_{3+14} g_4 = (u_{1+2})^{1/2} y_{3+14} \cdots\cdots\cdots\cdots (11),$

$(u_{1+2})^{1/2} g_3 = y_{3-14},\ (u_{1+2})^{1/2} g_4 = y_{1-2} \cdots\cdots (12).$

These equations (8)~(11) derived from the other equations (1)~(7), (12) as follows:

The equation (8) is

$$\begin{aligned}
&(u_{1+2})^{1/2} (u_{3-14} g_1 + u_{5-0} g_3) \\
&= (x_{3-14} y_{1-2} - x_{1-2} y_{3-14}) x_{1+2} + (x_{1+2} x_{1-2} - x_{3+14} x_{3-14}) y_{3-14} \\
&= (y_{1-2} x_{1+2} - x_{3+14} y_{3-14}) x_{3-14} \\
&= u_{1+2} x_{3-14}.
\end{aligned}$$

The equation (9) is

$$\begin{aligned}
&(u_{1+2})^{1/2} (u_{3-14} g_2 + u_{5-0} g_4) \\
&= (x_{3-14} y_{1-2} - x_{1-2} y_{3-14}) x_{3+14} + (x_{1+2} x_{1-2} - x_{3+14} x_{3-14}) y_{1-2} \\
&= (-y_{3-14} x_{3+14} + x_{1+2} x_{1-2}) x_{1-2} \\
&= u_{1+2} x_{1-2}.
\end{aligned}$$

The equation (10) is

$$\begin{aligned}
&(u_{1+2})^{1/2} (u_{5+0} g_1 - u_{3+14} g_3) \\
&= (y_{1+2} y_{1-2} - y_{3+14} y_{3-14}) x_{1+2} - (-x_{1+2} y_{3+14} + x_{3+14} y_{1+2}) y_{3-14} \\
&= (-y_{1-2} x_{1+2} - x_{3+14} y_{3-14}) y_{1+2} \\
&= u_{1+2} y_{1-2}.
\end{aligned}$$

The equation (11) is

$$\begin{aligned}
&(u_{1+2})^{1/2} (u_{5+0} g_2 - u_{3+14} g_4) \\
&= (y_{1+2} y_{1-2} - y_{3+14} y_{3-14}) x_{3+14} - (-x_{1+2} y_{3+14} + x_{3+14} y_{1+2}) y_{1-2} \\
&= (-y_{3-14} x_{3+14} + y_{1+2} y_{1-2}) y_{3+14} \\
&= u_{1+2} y_{3+14}.
\end{aligned}$$

(d),(e),(f) When $u_{1-2} \neq 0, u_{3+14} \neq 0$ and $u_{3-14} \neq 0$, the same manner as $u_{1+2} \neq 0$ is done. q.e.d.

§ 3. The transformation on the associated fiber bundle

Theorem 2.

The transformation group $SL(4,\mathbb{C})$ on $M_2(\mathbb{C})^2_0$ has a representation as the transformation group $SO(2,4,\mathbb{C})$ on $\mathbb{C}^{2,4}$ as follows:

if $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(4,\mathbb{C})$ then it corresponds to the element $h=\rho(g) \in SO(2,4,\mathbb{C})$ such that

$$h\cdot\begin{pmatrix} u_0 \\ U \\ u_5 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(|d|-|b|-|c|+|a|)u_0+\mathrm{Re}(cU\tilde{d}-aU\tilde{b})+\frac{1}{2}(|d|-|b|+|c|-|a|)u_5 \\ (b\tilde{d}-a\tilde{c})u_0+(aU\tilde{d}+b\tilde{U}\tilde{c})+(b\tilde{d}+a\tilde{c})u_5 \\ \frac{1}{2}(|d|+|b|-|c|-|a|)u_0+\mathrm{Re}(cU\tilde{d}+aU\tilde{b})+\frac{1}{2}(|d|+|b|+|c|+|a|)u_5 \end{pmatrix},$$

where we look upon the matrix $U=\begin{pmatrix} u_1+u_2 & u_3+iu_4 \\ u_3-iu_4 & u_1-u_2 \end{pmatrix}$ as the vector $\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$.

Moreover

(a) the subgroup

$$Sp'(2,\mathbb{C}) = \left\{ g=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(4,\mathbb{C}) \ / \ \tilde{g}\cdot g = E \right\},\ \tilde{g}=\begin{pmatrix} \tilde{a} & \tilde{c} \\ \tilde{b} & \tilde{d} \end{pmatrix}$$

corresponds to the $SO(2,3,\mathbb{C})$,

(b) the subgroup

$$Sp^*(2,\mathbb{C}) = \left\{ g=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(4,\mathbb{C}) \ / \ {}^t\bar{g}\cdot J_4\cdot g = J_4 \right\},\ J_4=\begin{pmatrix} 0 & E_2 \\ -E_2 & 0 \end{pmatrix}$$

corresponds to the $SO(2,4)$,

(c) the subgroup

$$\begin{aligned} Sp^*_0(2,\mathbb{C}) &= Sp'(2,\mathbb{C}) \cap Sp^*(2,\mathbb{C}) \\ &= \left\{ g=\begin{pmatrix} a & b \\ -{}^t\bar{\bar{b}} & {}^t\bar{\bar{a}} \end{pmatrix} \in SL(4,\mathbb{C}) \ / \ \tilde{g}\cdot g = E \right\} \end{aligned}$$

corresponds to the $SO(2,3)$,

(d) the subgroup

$$SL(2,\mathbb{C})\times SL(2,\mathbb{C}) = \left\{ g=\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in SL(4,\mathbb{C}) \ / \ a,d \in SL(2,\mathbb{C}) \right\}$$

corresponds to the $SO(1,3,\mathbb{C})$.

(e) the subgroup

$$SL'(2,\mathbb{C}) = \left\{ g=\begin{pmatrix} a & 0 \\ 0 & {}^t\bar{a}^{-1} \end{pmatrix} \in SL(4,\mathbb{C}) \ / \ a \in SL(2,\mathbb{C}) \right\}$$

corresponds to the $SO(1,3)$.

Proof

Let the transformation g on $M_2(\mathbb{C})^2_0$ be

$$SL(4,\mathbb{C}) \ni g=\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} aX+bY \\ cX+dY \end{pmatrix},\quad a,b,c,d,X,Y \in M_2(\mathbb{C})$$

then the transformation h on the base space $\tilde{S}^4(\mathbb{C})$ is as follows:

$$h:\begin{pmatrix} u_0 \\ U \\ u_5 \end{pmatrix} = \begin{pmatrix} |Y|-|X| \\ 2X\tilde{Y} \\ |Y|+|X| \end{pmatrix} \to \begin{pmatrix} |cX+dY|-|aX+bY| \\ 2(aX+bY)(cX\tilde{+}dY) \\ |cX+dY|+|aX+bY| \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}(|c|-|a|)(u_5-u_0)+\mathrm{Re}(cU\tilde{d}-aU\tilde{b})+\frac{1}{2}(|d|-|b|)(u_5+u_0) \\ a(u_5-u_0)\tilde{c}+\ (aU\tilde{d}+b\tilde{U}\tilde{c})+b(u_5+u_0)\tilde{d} \\ \frac{1}{2}(|c|+|a|)(u_5-u_0)+\mathrm{Re}(cU\tilde{d}+aU\tilde{b})+\frac{1}{2}(|d|+|b|)(u_5+u_0) \end{pmatrix}.$$

Therefore this transformation is the element of the $SO(2,4,\mathbb{C})$ because the equation

$(u_0)^2+|U|=(u_5)^2$, i.e., $(u_0)^2+(u_1)^2-(u_2)^2-(u_3)^2-(u_4)^2=(u_5)^2$

is invariant under the element h.

Therefore the equation ${}^t h I_{2,4} h = I_{2,4} \cdots (1)$, $I_{2,4}=\begin{pmatrix} -E_2 & 0 \\ 0 & E_4 \end{pmatrix}$ holds.

(a) Subgroup $Sp'(2,\mathbb{C})$ corresponds to the $SO(2,3,\mathbb{C})$,
because $\tilde{g}\cdot g = E$, i.e., $|a|+|c|=|b|+|d|=1$, $\tilde{a}b+\tilde{c}d=0$,
and $\mathrm{Re}(cU\tilde{d}+aU\tilde{b}) = \mathrm{Re}(\tilde{d}cU+\tilde{b}aU)$.
Therefore the following equation
$(u_0)^2+|U|=1$, i.e., $(u_0)^2+(u_1)^2-(u_2)^2-(u_3)^2-(u_4)^2=1$.
is invariant under the element h.

(b) Subgroup $Sp^*(2,\mathbb{C})$ corresponds to the $SO(2,4)$,
because if $g \in SL(4,\mathbb{C})$ then ${}^t\bar{g}$ and J_4 each correspond to ${}^t\bar{h}$ and $I_{2,4}$ by the direct calculation.
Therefore the equation ${}^t\bar{g}\, J_4 g = J_4$ correspond to the equation ${}^t\bar{h} I_{2,4} h = I_{2,4}$
and by equation (1) $\bar{h} = h$ holds.

(c) Subgroup $Sp^*_0(2,\mathbb{C})$ clearly corresponds to the $SO(2,3)$.

(d) Subgroup $SL(2,\mathbb{C})\times SL(2,\mathbb{C})$ corresponds to the $SO(1,3,\mathbb{C})$,
because $2(aU\tilde{d}+b\tilde{U}\tilde{c}) = 2(aU\tilde{d})$.

(e) Subgroup $SL'(2,\mathbb{C})$ corresponds to the $SO(1,3)$,
because $2(aU\tilde{d}+b\tilde{U}\tilde{c}) = 2(aU{}^t\bar{a})$. q.e.d.

We discuss the transformation of $Sp'(2,\mathbb{C})$ on the principal fiber bundle $P(S^7(\mathbb{C}),\pi,S^4(\mathbb{C}))$

Let $S^7(\mathbb{C}) = \left\{ \begin{pmatrix} X \\ Y \end{pmatrix} / \ |X|+|Y|=1 \right\}$

then the principal fiber bundle

$$\begin{array}{c} S^7(\mathbb{C}) \\ \pi \Big\downarrow \quad R_g : g \in SL(2,\mathbb{C}), \\ S^4(\mathbb{C}) \end{array}$$

is invariant under $Sp'(2,\mathbb{C})$
because

$$Sp'(2,\mathbb{C}) \ni g=\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} aX+bY \\ cX+dY \end{pmatrix},$$

and

$$|aX+bY|+|cX+dY| = (|X||a|+\tilde{X}\tilde{a}bY+\tilde{Y}\tilde{b}aX+|Y||b|) + (|X||c|+\tilde{X}\tilde{c}dY+\tilde{Y}\tilde{d}cX+|Y||d|)$$
$$=|X|(|a|+|b|)+\tilde{X}(\tilde{a}b+\tilde{c}d)Y+\tilde{Y}(\tilde{b}a+\tilde{d}c)X+|Y|(|b|+|d|)$$
$$=|X|+|Y|.$$

Moreover the canonical connection $\Theta_p \begin{pmatrix} dX \\ dY \end{pmatrix} = \mathrm{Im}(\tilde{X}dX+\tilde{Y}dY)=(\tilde{X}\ \tilde{Y})\begin{pmatrix} dX \\ dY \end{pmatrix}$

$= < \begin{pmatrix} X \\ Y \end{pmatrix}, \begin{pmatrix} dX \\ dY \end{pmatrix} >$ is invariant under the group action of $Sp'(2.\mathbb{C})$.

because

$$< \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} > = < \begin{pmatrix} X \\ Y \end{pmatrix}, \begin{pmatrix} X \\ Y \end{pmatrix} >,$$

means that

$$\begin{pmatrix} \tilde{a} & \tilde{c} \\ \tilde{b} & \tilde{d} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = E \quad ,\text{i.e.}, \ \tilde{g}\cdot g = E.$$

The base space $S^4(\mathbb{C})$ is 4-dimentional space and its frame bundle is a one with 4-dimentional frame and the transformation group $Sp'(2,\mathbb{C})$ on the fiber bundle $S^7(\mathbb{C})$ act as the transformation group $SO(2,3,\mathbb{C})$.

In the real case, we use the transformation groups $Sp^*_0(2,\mathbb{C})$ and $SO(2,3)$ instead of the transformation groups $Sp'(2,\mathbb{C})$ and $SO(2,3,\mathbb{C})$.

Let the canonical 5-dimentional frame be A_1, A_2, A_3, A_4, N and the stereographic projection ϕ_{-1} on S^4 $(u_0 \neq -1)$ such that $\phi_{-1}:(u_0, U) \to u = Z(u_0+1)^{-1} \in \mathbb{C}^4(|u| \neq -1)$

Then

$$(\phi_{-1})^{-1}: u \to ((1-|u|)(1+|u|)^{-1}, 2u(1+|u|)^{-1})$$

and

$$\begin{aligned} d(\phi_{-1})^{-1}(u) = \ & 2(1+|u|)^{-2}(-2u^1, 1+|u|-2(u^1)^2, \ -2u^2u^1, \ -2u^3u^1, \ -2u^4u^1)du^1 \\ & +2(1+|u|)^{-2}(2u^2, \ 2u^1u^2, 1+|u|+2(u^2)^2, \ 2u^3u^2, \ 2u^4u^2)du^2 \\ & +2(1+|u|)^{-2}(2u^3, \ 2u^1u^3, \ 2u^2u^3, 1+|u|+2(u^3)^2, \ 2u^4u^3)du^3 \\ & +2(1+|u|)^{-2}(2u^4, \ 2u^1u^4, \ 2u^2u^4, \ 2u^2u^4, 1+|u|+2(u^4)^2)du^4. \end{aligned}$$

Therefore A_1, A_2, A_3, A_4 are each coefficient of the du^1, du^2, du^3, du^4 such that

$$d(\phi_{-1})^{-1}(u) = 2(1+|u|)^{-2}(A_1du^1+A_2du^2+A_3du^3+A_4du^4)$$

and N a vector which is orthogonal to A_1, A_2, A_3, A_4 (i.e. satisfy the condition $2u_0du_0+d|U|=0$) then $N = (1-|u|, 2u^1, 2u^2, 2u^3, 2u^4)$.

For the stereographic projection ϕ_1, the same manner is used as above.

Proposition 3.

The following A_1', A_2', A_3', A_4' and N' are orthonormal frame in $\mathbb{C}^{2,3}$

$$A_1' = (1+|u|)^{-1}A_1,\ A_2' = (1+|u|)^{-1}A_2,\ A_3' = (1+|u|)^{-1}A_3,\ A_4' = (1+|u|)^{-1}A_4,\ N' = (1+|u|)^{-1}N$$

and

$$N = \begin{pmatrix} 1-|u| \\ 2u^1 \\ 2u^2 \\ 2u^3 \\ 2u^4 \end{pmatrix}\quad A_1 = \begin{pmatrix} -2u^1 \\ 1+|u|-2(u^1)^2 \\ -2u^2u^1 \\ -2u^3u^1 \\ -2u^4u^1 \end{pmatrix}\quad A_2 = \begin{pmatrix} 2u^2 \\ 2u^1u^2 \\ 1+|u|+2(u^2)^2 \\ 2u^3u^2 \\ 2u^4u^2 \end{pmatrix}\quad A_3 = \begin{pmatrix} 2u^3 \\ 2u^1u^3 \\ 2u^2u^3 \\ 1+|u|+2(u^3)^2 \\ 2u^4u^3 \end{pmatrix}\quad A_4 = \begin{pmatrix} 2u^4 \\ 2u^1u^4 \\ 2u^2u^4 \\ 2u^3u^4 \\ 1+|u|+2(u^4)^2 \end{pmatrix}$$

Proposition 4.

A canonical connection on the 5-dimentional frame bundle on $S^4(\mathbb{C})$ is

$$\tilde{A}'(u) = \frac{2}{1+|u|}\left(\begin{array}{cc|ccc} 0 & -du^1 & du^2 & du^3 & du^4 \\ du^1 & 0 & u^2du^1-u^1du^2 & u^3du^1-u^1du^3 & u^4du^1-u^1du^4 \\ \hline du^2 & u^2du^1-u^1du^2 & 0 & u^3du^2-u^2du^3 & u^4du^2-u^2du^4 \\ du^3 & u^3du^1-u^1du^3 & u^2du^3-u^3du^2 & 0 & u^4du^3-u^3du^4 \\ du^4 & u^4du^1-u^1du^4 & u^2du^4-u^4du^2 & u^3du^4-u^4du^3 & 0 \end{array}\right) \in so(2,3,\mathbb{C})$$

and a canonical connection (i.e., Levi Civita connection) on the base space $S^4(\mathbb{C})$ is

$$A'(u) = \frac{2}{1+|u|}\left(\begin{array}{c|ccc} 0 & u^2du^1-u^1du^2 & u^3du^1-u^1du^3 & u^4du^1-u^1du^4 \\ \hline u^2du^1-u^1du^2 & 0 & u^3du^2-u^2du^3 & u^4du^2-u^2du^4 \\ u^3du^1-u^1du^3 & u^2du^3-u^3du^2 & 0 & u^4du^3-u^3du^4 \\ u^4du^1-u^1du^4 & u^2du^4-u^4du^2 & u^3du^4-u^4du^3 & 0 \end{array}\right) \in so(1,3,\mathbb{C})$$

Proof

Let

$$h = (1+|u|)^{-1}\begin{pmatrix} 1-|u|, & -2u^1, & 2u^2, & 2u^3, & 2u^4 \\ 2u^1, & 1+|u|-2(u^1)^2, & 2u^1u^2, & 2u^1u^3, & 2u^1u^4 \\ 2u^2, & -2u^2u^1, & 1+|u|+2(u^2)^2, & 2u^2u^3, & 2u^2u^4 \\ 2u^3, & -2u^3u^1, & 2u^3u^2, & 1+|u|+2(u^3)^2, & 2u^3u^4 \\ 2u^4, & -2u^4u^1, & 2u^4u^2, & 2u^4u^3, & 1+|u|+2(u^4)^2 \end{pmatrix}$$

then

$$\tilde{A}'(u) = h^{-1}\cdot dh$$

and A'(u) is obtained from $\tilde{A}'(u)$ by the projection $so(2,3.\mathbb{C}) \to so(1,3.\mathbb{C})$. q.e.d.

§4. The decomposition of SL(4, $\mathbb{C}$) and crosssection

We discuss the transformation of SL(4, $\mathbb{C}$) on the principal fiber bundle $P(M_2(\mathbb{C})^2_0, \pi, S^4(\mathbb{C}))$.

Let $a, b, c, d, X, Y \in M_2(\mathbb{C})$

$$SL(4,\mathbb{C}) \ni g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} aX+bY \\ cX+dY \end{pmatrix} \text{ on } M_2(\mathbb{C})^2$$

Then the canonical connection $\Theta_p \begin{pmatrix} dX \\ dY \end{pmatrix} = \tilde{X}dX + \tilde{Y}dY = (\tilde{X}, \tilde{Y})\begin{pmatrix} dX \\ dY \end{pmatrix}$ is invariant under the subgroup $Sp'(2,\mathbb{C})$ of $SL(4,\mathbb{C})$.

The fiber space is divided into two parts as follows:

$$M_2(\mathbb{C})^2_0 = M_2(\mathbb{C})^2_{-1} \cup M_2(\mathbb{C})^2_1$$
$$M_2(\mathbb{C})^2_{-1} = M_2(\mathbb{C})^2_0 \cap \{|Y| \neq 0\},$$
$$M_2(\mathbb{C})^2_1 = M_2(\mathbb{C})^2_0 \cap \{|X| \neq 0\}.$$

and on the each part

$N_{-1} = \begin{pmatrix} E & n \\ 0 & E \end{pmatrix}$ is transitive on the base space of $M_2(\mathbb{C})^2_{-1}$, because $N_{-1}\begin{pmatrix} 0 \\ E \end{pmatrix} = \begin{pmatrix} n \\ E \end{pmatrix}$,

$N_1 = \begin{pmatrix} 0 & E \\ E & \tilde{n} \end{pmatrix}$ is transitive on the base space of $M_2(\mathbb{C})^2_1$ because $N_1\begin{pmatrix} E \\ 0 \end{pmatrix} = \begin{pmatrix} E \\ \tilde{n} \end{pmatrix}$.

Proposition 5.

The transformation group $SL(4, \mathbb{C})$ $(|b|+|d| \neq 0)$ has a decomposition $SP'(2, \mathbb{C}) \cdot A \cdot N$ as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} (|b|+|d|)^{-1/2}(a|d|-b\tilde{d}c) & (|b|+|d|)^{-1/2}b \\ (|b|+|d|)^{-1/2}(c|b|-d\tilde{b}a) & (|b|+|d|)^{-1/2}d \end{pmatrix} \begin{pmatrix} (|b|+|d|)^{-1/2}E & 0 \\ 0 & (|b|+|d|)^{1/2}E \end{pmatrix} \begin{pmatrix} E & 0 \\ (|b|+|d|)^{-1}(\tilde{b}a+\tilde{d}c) & E \end{pmatrix}$$

Proof

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} tE & 0 \\ 0 & t^{-1}E \end{pmatrix} \begin{pmatrix} E & 0 \\ n & E \end{pmatrix}$ $\quad \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in Sp'(2, \mathbb{C})$, $t(\neq 0) \in \mathbb{C}$, $n \in M_2(\mathbb{C})$

$$= \begin{pmatrix} xt+yt^{-1}n & yt^{-1} \\ zt+wt^{-1}n & wt^{-1} \end{pmatrix}.$$

Therefore

$$a = xt+yt^{-1}n = xt+bn, \qquad b = yt^{-1},$$
$$c = zt+wt^{-1}n = zt+dn, \qquad d = wt^{-1},$$

and

$$|b|+|d| = (|y|+|w|)t^{-2} = t^{-2},$$

then

$$t = (|b|+|d|)^{-1/2} (\neq 0) \in \mathbb{C},$$
$$y = tb = (|b|+|d|)^{-1/2}b,$$
$$w = td = (|b|+|d|)^{-1/2}d \quad \text{and } |y|+|w| = 1.$$

And

$$\tilde{a}b+\tilde{c}d = (t\tilde{x}+\tilde{n}\tilde{b})yt^{-1} + (t\tilde{z}+\tilde{n}\tilde{d})wt^{-1} = \tilde{n}(|b|+|d|),$$

then

$$n = (|b|+|d|)^{-1}(\tilde{b}a+\tilde{d}c) \in M_2(\mathbb{C}),$$
$$x = (|b|+|d|)^{-1/2}(a|d|-b\tilde{d}c),$$
$$z = (|b|+|d|)^{-1/2}(c|b|-d\tilde{b}a).$$

$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in Sp'(2, \mathbb{C})$ hold as follows:

1) $\tilde{x}y + \tilde{z}w = (|b|+|d|)^{-1}(|d|\tilde{a}b - |b|\tilde{c}d) + (|b|+|d|)^{-1}(|b|\tilde{c}d - |d|\tilde{a}b) = 0,$

2) $|y|+|w| = (|b|+|d|)^{-1}(|b|+|d|) = 1,$

3) $|x|+|z| = \frac{1}{2}(|b|+|d|)^{-1}\, \mathrm{Tr}\, \{(\tilde{a}|d|-\tilde{c}d\tilde{b})(a|d|-b\tilde{d}c) + (\tilde{c}|b|-\tilde{a}b\tilde{d})(c|b|-d\tilde{b}a)\}$

$$= (|b|+|d|)^{-1}\{|a||d|^2 - \mathrm{Tr}(\tilde{a}b\tilde{d}c)|d| + |b||d||c| + |c||b|^2 - \mathrm{Tr}(\tilde{a}b\tilde{d}c)|b| + |d||b||a|)$$
$$= |a||d| - \mathrm{Tr}(\tilde{a}b\tilde{d}c) + |b||c|$$

$$= \mathrm{Det}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \because \text{ proposition 6 below.}$$

$$= 1.$$

The uniqueness of the decomposition is as follows:

If $\begin{pmatrix} x & y \\ z & w \end{pmatrix}\begin{pmatrix} tE & 0 \\ 0 & t^{-1}E \end{pmatrix}\begin{pmatrix} E & 0 \\ n & E \end{pmatrix} = \begin{pmatrix} x' & y' \\ z' & w' \end{pmatrix}\begin{pmatrix} t'E & 0 \\ 0 & t'^{-1}E \end{pmatrix}\begin{pmatrix} E & 0 \\ n' & E \end{pmatrix}$,

then $\begin{pmatrix} x' & y' \\ z' & w' \end{pmatrix}^{-1}\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} t'E & 0 \\ 0 & t'^{-1}E \end{pmatrix}\begin{pmatrix} E & 0 \\ n'-n & E \end{pmatrix}\begin{pmatrix} t^{-1}E & 0 \\ 0 & tE \end{pmatrix} = \begin{pmatrix} t't^{-1}E & 0 \\ t'^{-1}t^{-1}(n'-n) & t'^{-1}tE \end{pmatrix} \in SP'(2, \mathbb{C})$

Therefore

$t'^{-1}t=1,\ t'^{-2}(\tilde{n}'-\tilde{n})=0 \quad \therefore t'=t, n'=n \quad \therefore x'=x,\ y'=y,\ z'=z,\ w'=w.$ q.e.d.

proposition 6.

Let $a,b,c,d \in M_2(\mathbb{C})$ then

$$\mathrm{Det}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = |a|\,|d| + |c|\,|b| - \mathrm{Tr}(\tilde{a}b\tilde{d}c)$$

hold

Proof

We check the following property (Weierstrass-Kronecker) of the right side:

(i) There is a linearity with respect to each column and row,

(ii) If adjacent columns or row are equal then the value is equal to 0.

For (i)

If $a = \begin{pmatrix} ka_1+k'a_1{'}a_2 \\ ka_3+k'a_3{'}a_4 \end{pmatrix}$ and $c = \begin{pmatrix} kc_1+k'c_1{'}c_2 \\ kc_3+k'c_3{'}c_4 \end{pmatrix}$ then

$\mathrm{Tr}(\tilde{a}b\tilde{d}c) = \mathrm{Tr}(c\tilde{a}b\tilde{d})$

and $c\tilde{a} = \begin{pmatrix} kc_1+k'c_1{'}c_2 \\ kc_3+k'c_3{'}c_4 \end{pmatrix}\begin{pmatrix} ka_1+\tilde{k}'a_1{'}a_2 \\ ka_3+k'a_3{'}a_4 \end{pmatrix}$

$$= \begin{pmatrix} (kc_1+k'c_1{'})a_4-c_2(ka_3+k'a_3{'}) & -(kc_1+k'c_1{'})a_2-c_2(ka_1+k'a_1{'}) \\ (kc_3+k'c_3{'})a_4-c_4(ka_3+k'a_3{'}) & -(kc_3+k'c_3{'})a_2-c_4(ka_1+k'a_1{'}) \end{pmatrix}$$

$$= k\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} + k'\begin{pmatrix} c_1{'} & c_2 \\ c_3{'} & c_4 \end{pmatrix}\begin{pmatrix} a_1{'} & a_2 \\ a_3{'} & a_4 \end{pmatrix}.$$

This means that right side is linear for the first column and the same statement holds for the another column and row.

For (ii)

If $a = \begin{pmatrix} a_1 & a_1 \\ a_3 & a_3 \end{pmatrix}$ and $c = \begin{pmatrix} c_1 & c_1 \\ c_3 & c_3 \end{pmatrix}$ then

$\mathrm{Tr}(\tilde{a}b\tilde{d}c) = \mathrm{Tr}(c\tilde{a}b\tilde{d}) = 0$

and $c\tilde{a} = \begin{pmatrix} c_1 & c_1 \\ c_3 & c_3 \end{pmatrix}\begin{pmatrix} a_1 & a_1 \\ a_3 & a_3 \end{pmatrix} = \begin{pmatrix} c_1a_3-c_1a_3 & -c_1a_1+c_1a_1 \\ c_3a_3-c_3a_3 & -c_3a_1+c_3a_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$

If $a = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, $b = \begin{pmatrix} a_2 & b_2 \\ a_4 & b_4 \end{pmatrix}$, $c = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$ and $d = \begin{pmatrix} c_2 & d_2 \\ c_4 & d_4 \end{pmatrix}$ then

$$\tilde{a}b\tilde{d}c) = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ a_4 & b_4 \end{pmatrix} \begin{pmatrix} c_2 & d_2 \\ c_4 & d_4 \end{pmatrix} \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & a_4b_2-a_2b_4 \\ -a_3a_2+a_1a_4 & -a_3b_2+a_1b_4 \end{pmatrix} \begin{pmatrix} d_4c_1-d_2c_3 & d_4c_2-d_2c_4 \\ -c_4c_1+c_2c_3 & 0 \end{pmatrix},$$

$$\mathrm{Tr}(\tilde{a}b\tilde{d}c) = (a_4b_2-a_2b_4)(-c_4c_1+c_2c_3)+(-a_3a_2+a_1a_4)(d_4c_2-d_2c_4)$$
$$= |b||c|+|a||d|.$$

This means that right side is zero when the first and the second/the second and the third column is equai.

The same statement holds when another adjacent columns or row are equal. q.e.d.

Theorem 7.
Let new extended Hopf fiber bundle and its associated bundle be

$$\begin{array}{ccc} & & Sp'(2,\mathbb{C}) \\ & \overset{\mu_{\pm1}}{\nearrow} & \downarrow \\ \mathbb{C}^4 & \longrightarrow & Sp'(2,\mathbb{C})/\begin{pmatrix} SL(2,\mathbb{C}) & 0 \\ 0 & SL(2,\mathbb{C}) \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{c} SO(2,3,\mathbb{C}) \\ \downarrow \\ SO(2,3,\mathbb{C})/SO(1,3,\mathbb{C}). \end{array}$$

Then there is a correspondence of (Levi−Civita) connections between above two bundles under the following locally isomorphism in theorem 2

$$\rho : Sp'(2,\mathbb{C}) \quad \to \quad SO(2,3,\mathbb{C}).$$

Proof by proposition 5, the cross section $\mu_{\pm1}:\mathbb{C}^4\to Sp'(2,\mathbb{C})$ is as follows:

$$\begin{pmatrix} E & u \\ 0 & E \end{pmatrix} = \begin{pmatrix} (|u|+1)^{-1/2}E & (|u|+1)^{-1/2}u \\ -(|u|+1)^{-1/2}\tilde{u} & (|u|+1)^{-1/2}E \end{pmatrix} \begin{pmatrix} (|u|+1)^{-1/2}E & 0 \\ 0 & (|u|+1)^{1/2}E \end{pmatrix} \begin{pmatrix} E & 0 \\ (|u|+1)^{-1}\tilde{u} & E \end{pmatrix},$$

$$\begin{pmatrix} 0 & E \\ E & \tilde{u}' \end{pmatrix} = \begin{pmatrix} -(|u'|+1)^{-1/2}u' & (|n'|+1)^{-1/2}E \\ (|u'|+1)^{-1/2}E & (|n'|+1)^{-1/2}\tilde{u}' \end{pmatrix} \begin{pmatrix} (|u'|+1)^{-1/2}E & 0 \\ 0 & (|u'|+1)^{1/2}E \end{pmatrix} \begin{pmatrix} E & 0 \\ (|u'|+1)^{-1}u' & E \end{pmatrix}.$$

Therefore the each cross section is

$$\mathbb{C}^4(|u|\neq-1)\ni u\to\mu_{-1}(u) = \begin{pmatrix} (|u|+1)^{-1/2}E & (|u|+1)^{-1/2}u \\ -(|u|+1)^{-1/2}\tilde{u} & (|u|+1)^{-1/2}E \end{pmatrix},$$

$$\mathbb{C}^4(|u'|\neq-1)\ni u'\to\mu_{1}(u') = \begin{pmatrix} -|u'|+1)^{-1/2}u' & (|u'|+1)^{-1/2}E \\ (|u'|+1)^{-1/2}E & (|u'|+1)^{-1/2}\tilde{u}' \end{pmatrix}.$$

The connection is as follows:

Let

$$g = \frac{1}{(1+|u|)^{1/2}} \begin{pmatrix} E & u \\ -\tilde{u} & E \end{pmatrix}$$

Then

$$A = g^{-1}\cdot dg$$
$$= \frac{1}{(1+|u|)^{1/2}} \begin{pmatrix} E & -u \\ \tilde{u} & E \end{pmatrix} \left(-\frac{1}{2}\frac{d|u|}{(1+|u|)^{3/2}} \begin{pmatrix} E & u \\ -\tilde{u} & E \end{pmatrix} + \frac{1}{(1+|u|)^{1/2}} \begin{pmatrix} 0 & du \\ -d\tilde{u} & 0 \end{pmatrix} \right)$$
$$= -\frac{1}{2}\frac{d|u|}{1+|u|} \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} + \frac{1}{1+|u|} \begin{pmatrix} ud\tilde{u} & du \\ -d\tilde{u} & \tilde{u}du \end{pmatrix}$$

$$= \frac{1}{1+|u|}\begin{pmatrix} \mathrm{Im}\, u d\tilde{u} & du \\ d\tilde{u} & \mathrm{Im}\, \tilde{u} du \end{pmatrix}.$$

this connection is correspond to the $\tilde{A}'(u)$ (in proposition 4) under the locally isomorphism ρ.
For the $\mu_1(u')$ the same manner as $\mu_{-1}(u)$ is done.

the connection A'(u) on the base space is obtained from A by the projection $sp'(2, \mathbb{C}) \to sl(2, \mathbb{C}) \times sl(2, \mathbb{C})$

q.e.d.

References

(1) S.Helgason Differential Geometry, Lie Groups, and Symmetric Spaces Academic Press,Inc. (1978)
(2) Kobayashi and Nomizu Foundations of Differential Geometry I Interscience Publishers(1969)
(3) Y.Takemoto "Pseudo"-Fiber Bundle and Connection on it Bull.of NBU Vol.20,No.1(1992-Feb.)