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Abstract

In this paper, we discuss the Vector Analysis on a Time-Space and show a existance of the potential which is supposed in the previous paper[4].

In this process, we use the potential and field of the 4-dimensional form. Then various formula in the 3-dimensional space is expressed as a part of the formula in the 4-dimensional space.

Contents :

In § 1 we review a traditional Gauss and Green's theorem in 3 dimensional space and their examples.

In § 2 we study Gauss' and Green's theorem in 4-dimensional time-space.

In § 3 we study Laplace-Poisson's equation in 4-dimensional time-space.

In § 4 we study Helmholtz' theorem in 4-dimensional time-space.

§ 1. Vector Analysis on the 3 dimensional Space

In this section, we give some theorem (i. e., Gauss', Green's and Helmholz' theorem) and it's exmples (i. e., the solution of Laplace-Poisson's equation and Maxwell's equation) of the 3-dimensional case.

Let's $dV = dx\wedge dy\wedge dz$, $d\mathbb{S} = (dy\wedge dz,\ dz\wedge dx,\ dx\wedge dy)$, D : domain, $S = \partial D$: surface of D.

[Theorem 1.] (Gauss' theorem)

Let's $\mathbb{A}$, ϕ be vector and scalar field then

$$\int_D \operatorname{div} \mathbb{A}\ dV = \int_S \mathbb{A}\cdot d\mathbb{S} = \int_S d\mathbb{S}\cdot\mathbb{A}, \text{ where "}\cdot\text{" is a scalar product.} \qquad \cdots (1)$$

$$\because\quad (\partial_x A_x + \partial_y A_y + \partial_z A_z)\, dx\wedge dy\wedge dz = A_x dy\wedge dz + A_y dz\wedge dx + A_z dx\wedge dy.$$

$$\int_D \operatorname{rot} \mathbb{A}\ dV = -\int_S \mathbb{A}\times d\mathbb{S} = \int_S d\mathbb{S}\times\mathbb{A}, \text{ where "}\times\text{" is a vector product.} \qquad \cdots (2)$$

$$\because\quad (\partial_y A_z - \partial_z A_y)\, dx\wedge dy\wedge dz = -(A_y dx\wedge dy - A_z dz\wedge dx)$$

$$(\partial_z A_x - \partial_x A_z)\, dx\wedge dy\wedge dz = -(A_z dy\wedge dz - A_x dx\wedge dy)$$

$$(\partial_x A_y - \partial_y A_x)\, dx\wedge dy\wedge dz = -(A_x dz\wedge dx - A_y dy\wedge dz).$$

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$$\int_D \mathrm{grad}\phi\, dV = \int_S \phi\, d\mathbb{S} = \int_S d\mathbb{S}\,\phi \qquad \cdots (3)$$

$\because \quad (\partial_x\phi,\ \partial_y\phi,\ \partial_z\phi)\, dx\Lambda dy\Lambda dz = (\phi dy\Lambda dz,\ \phi dz\Lambda dx, \phi dx\Lambda dy).$

[Theorem 2.] (Green's theorem)

Let's ϕ, Ψ be scalar fields then

$$\int_S \Psi\ \mathrm{grad}\ \phi\cdot d\mathbb{S} = \int_D (\mathrm{grad}\ \Psi\cdot \mathrm{grad}\ \phi + \Psi\Delta\phi)\, dV \qquad \cdots (4)$$

$$\int_S \Psi\ \mathrm{grad}\ \phi - \phi\ \mathrm{grad}\ \Psi)\cdot d\mathbb{S} = \int_D (\Psi\Delta\phi - \phi\Delta\Psi)\, dV \qquad \cdots (5)$$

$\because \mathrm{div}(\Psi\ \mathrm{grad}\ \phi) = \mathrm{grad}\ \Psi\cdot\mathrm{grad}\ \phi + \Psi\ \mathrm{div\ grad}\ \phi$

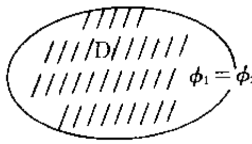
$\qquad = \mathrm{grad}\ \Psi\cdot\mathrm{grad}\ \phi + \Psi\Delta\phi,\ \ \Delta\phi \equiv \mathrm{div\ grad}\ \phi$

and by (1), Gauss' theorem, formula (4) and (5) holds.

[Example 3.] An uniqueness of solution of Laplace-Poisson's equation $(\Delta\phi = -4\pi\rho)$.

Let's ϕ_1, ϕ_2 be the solution of Laplace-Poisson's equation $\Delta\phi = -4\pi\rho$ in the domain D, and we adapt $\phi = \Psi = \phi_1 - \phi_2$ to the Green's theorem (4), then

$$\int_D \mathrm{grad}^2\ (\phi_1 - \phi_2)\ dV = \int_S\ (\phi_1 - \phi_2)\ \mathrm{grad}\ (\phi_1 - \phi_2)\cdot d\mathbb{S}.$$



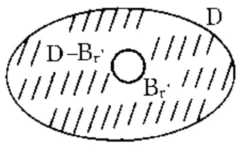
$\phi_1 = \phi_2$ on S $= \partial D$

If $\phi_1 = \phi_2$ in $S = \partial D$ (surface of D), then

$\mathrm{grad}(\phi_1 - \phi_2) = 0$ in D, and $\phi_1 = \phi_2$ in D.

[Example 4.] An property of solution of Laplace-Poisson's equation $(\Delta\phi = -4\pi\rho)$.

Let's $B_{r'}$ be a ball of radius r' and $S_{r'} = \partial B_{r'}$ a sphere of $B_{r'}$, and $\Psi = 1/r$, then $\Delta\Psi = 0$ on $D - B_{r'}$.



Then by (5) (Green's theorem)

$$\int_{S-S_{r'}} ((1/r)\mathrm{grad}\phi - \phi\mathrm{grad}(1/r))\cdot d\mathbb{S} = \int_{D-B_{r'}} (\Delta\phi)/r\ dV,$$

therfore

$$-\int_{S_{r'}} \phi\mathrm{grad}(1/r)\cdot d\mathbb{S} = -\int_{D-B_{r'}} (\Delta\phi)/r\ dV - \int_S \phi\mathrm{grad}(1/r)\cdot d\mathbb{S} + \int_S (1/r)\mathrm{grad}\phi\cdot d\mathbb{S} - \int_{S_{r'}} (1/r)\mathrm{grad}\phi\cdot d\mathbb{S}.$$

When $r' \to 0$

$$4\pi\phi(0) = 4\pi \int_D \rho/r\ dV + 1/r^2 \int_S \phi \mathbb{r}/r\cdot d\mathbb{S} + 1/r \int_S \mathrm{grad}\ \phi\ \cdot d\mathbb{S}. \qquad \cdots (6)$$

Moreover let $D = B_r$, $S = S_r$ and $\Delta\phi = -4\pi\rho$ then at the point $\rho = 0$,

$$\int_{S_r} \mathrm{grad}\ \phi\cdot d\mathbb{S} = \int_{B_r} \mathrm{div\ grad}\ \phi\ dV = -4\pi \int_{B_r} \rho\ dv = 0,$$

and

$$\phi(0) = 1/r^2 \int_{S_r} \phi \mathbb{r}/r\cdot d\mathbb{S}$$

, i. e., the value of $\phi(0)$ is decided by the only values on the sphere S_r.
Conversely, the following potential

$$\phi(x', y', z') = \int_{V} \rho(x, y, z)/r(x-x', y-y', z-z')\,dV$$
V : whole space

satisfies the Laplace poisson's equation $\Delta\phi = -4\pi\rho$.
Because

$$\begin{aligned}
\int_{B_{r'}} \Delta\phi\, dV' &= \int_{B_{r'}} \Big[\Delta \int_{V\cap B_{r'}} (\rho/r)dV + \Delta \int_{V\setminus B_{r'}} (\rho/r)dV\Big]\, dV' \\
&= \int_{B_{r'}} \Big(\mathrm{div} \int_{B_{r'}} \mathrm{grad}(\rho/r)dV\Big)dV' \qquad \because \Delta_{p'} \int_{V\setminus B_{r'}} (\rho_p/r_{p,p'})dV_p = 0 \text{ on } B_{r',p'} \\
&= -\int_{S_{r'}} \Big(\int_{B_{r'}} \rho\, \mathbb{r}/r^3 dV\Big)\cdot d\mathbb{S}' \\
&= -\int_{B_{r'}} \Big(\int_{S_{r'}} \rho/r^2(\mathbb{r}/r)\cdot d\mathbb{S}'\Big)dV \\
&= -4\pi \int_{B_{r'}} \rho\, dV,
\end{aligned}$$

and r' is arbitrary.

[Helmholtz' theorem 5.]
An arbitrary vector field $\mathbb{E}$ is represented as the sum of two vector fields gradϕ and rot A as follows :

$$\mathbb{E} = -\mathrm{grad}\phi - \mathrm{rot}\ \mathbb{A}$$

where ϕ is a scalar potential and $\mathbb{A}$ is a 3 dimensional vector potential.
Especially following relation are hold.
i) there is only a scalar potential ϕ such that $\mathbb{E} = -\mathrm{grad}\ \phi$ iff rot $\mathbb{E} = 0$,
ii) there is only a vector potential $\mathbb{A}$ such that $\mathbb{E} =\ \ \mathrm{rot}\ \ \mathbb{A}$ iff div $\mathbb{E} = 0$.
Because
Let's $\mathbb{E} = (E_x, E_y, E_z)$, then each $E^0{}_i = \int_V (E_i/r)\, dV\ (i = x, y, z)$ satisfies the Laplace-Poisson's equation $\Delta E^0{}_i = -4\pi E_i$ by the above example 4,
and

$$\begin{aligned}
\mathbb{E} &= -\Delta \int_V (\mathbb{E}/4\pi r)dv, \\
&= -\mathrm{grad} \int_V (\mathrm{div}\ \mathbb{E}/4\pi r)dv + \mathrm{rot} \int_V (\mathrm{rot}\ \mathbb{E}/4\pi r)dv.
\end{aligned}$$

Therefore we define a scalar potential ϕ and vector potential $\mathbb{A}$ as follows :

$$\phi = \int_V (\mathrm{div}\ \mathbb{E})/4\pi r\ dV + \underline{\mathrm{div}\chi},$$
V : whole space

$$\mathbb{A} = \int_V (\mathrm{rot}\ \mathbb{E})/4\pi r\ dV + \mathrm{grad}\chi_0 + \underline{\mathrm{rot}\ \chi},$$
where χ_0 is an arbitrary continuously differentiable function and χ satisfy $\Delta\chi = 0$.

[Example 6.] (Potential of electromagnetic fields)

A electromagnetic fields $\mathbb{E}$ and $\mathbb{B}$ satisfy the following equation :

$$\begin{cases} \mathrm{rot}\ \mathbb{E}+\partial\mathbb{B}/\partial t=0 & \cdots (6)\\ \mathrm{div}\ \mathbb{B}=0 & \cdots (7)\\ \mathrm{div}\ \mathbb{E}=\rho & \cdots (8)\\ \mathrm{rot}\ \mathbb{B}-\partial\mathbb{E}/\partial t=\mathbb{J} & \cdots (9)\end{cases}$$

From equation (7), there exists a vector potential $\mathbb{A}$ such that $\mathbb{B}=\mathrm{rot}\ \mathbb{A}$ and from this and equation (6), $\mathrm{rot}(\mathbb{E}+\partial\mathbb{A}/\partial t)=0$.
Therefore there exists a scalar potential ϕ such that $\mathbb{E}+\partial\mathbb{A}/\partial t=-\mathrm{grad}\ \phi$

and equations (8), (9) yield to

$$\begin{aligned}\rho&=\mathrm{div}\ \mathbb{E}\\ &=-\mathrm{div}(\partial\mathbb{A}/\partial t+\mathrm{grad}\ \phi)\\ &=-\partial(\mathrm{div}\ \mathbb{A})/\partial t-\Delta\phi\\ &=-\partial/\partial t(\mathrm{div}\ \mathbb{A}+\partial\phi/\partial t)-\square\ \phi,\end{aligned}$$

$$\square\ \phi=-\ [\rho+\underline{\partial/\partial t(\mathrm{div}\ \mathbb{A}+\partial\phi/\partial t)}]\ .$$

$$\begin{aligned}\mathbb{J}&=\mathrm{rot}\ \mathbb{B}-\partial\mathbb{E}/\partial t\\ &=\mathrm{rot}(\mathrm{rot}\ \mathbb{A})+\partial(\partial\mathbb{A}/\partial t+\mathrm{grad}\ \phi)/\partial t\\ &=-\Delta\mathbb{A}+\mathrm{grad}(\mathrm{div}\ \mathbb{A})+\partial^2\mathbb{A}/\partial t^2+\partial(\mathrm{grad}\ \phi)/\partial t\\ &=\mathrm{grad}(\mathrm{div}\ \mathbb{A}+\partial\phi/\partial t)-\square\mathbb{A},\end{aligned}$$

$$\square\mathbb{A}=-\ [\mathbb{J}-\underline{\mathrm{grad}(\mathrm{div}\ \mathbb{A}+\partial\phi/\partial t)}]\ .$$

where each underlined parts means charge $\partial E_t/\partial t$ and current $\mathrm{grad}E_t$ [4].

§2. Gauss' and Green's theorem on 4-dimensional time-space

Proposition 7. (4-dimensional Gauss' theorem)

A following formula is hold

$$\int_U DE\ dV\Lambda dct=\int_{\partial U}(dS)E,\quad U: \text{4-dimensional domain, } \partial U: \text{border of } U \qquad \cdots (10)$$

where

$$D=\begin{pmatrix}\partial_t & \\ & \nabla\end{pmatrix}=\begin{pmatrix}\partial_t+\partial_x & \partial_y+i\partial_z\\ \partial_y-i\partial_z & \partial_t-\partial_x\end{pmatrix},\quad \nabla=\partial_x i+\partial_y j+\partial_z k$$

is a 4-dimensional differential operator,

$$E=\begin{pmatrix}E_t & \\ & \mathbb{E}\end{pmatrix}=\begin{pmatrix}E_t+E_x & E_y+iE_z\\ E_y-iE_z & E_t-E_x\end{pmatrix},\quad \mathbb{E}=E_x i+E_y j+E_z k$$

is a 4-dimensional vector,

$$dS=\begin{pmatrix}dV & \\ & d\mathbb{S}\Lambda dct\end{pmatrix}=\begin{pmatrix}dV+dS_x & dS_y+idS_z\\ dS_y-idS_z & dV-dS_x\end{pmatrix},\quad d\mathbb{S}\Lambda dct=dS_x i+dS_y j+dS_z k$$

is a 4-dimensional surface element, and

$$dV\Lambda dct=dx\Lambda dy\Lambda dz\Lambda dct$$

is a 4-dimensional volume element

$$dV=dx\Lambda dy\Lambda dz$$

$$dS_x=dy\Lambda dz\Lambda dct,\ dS_y=dz\Lambda dx\Lambda dct,\ dS_z=dx\Lambda dy\Lambda dct.$$

And D, E, dS(matrix) and dVΛdct(scalar) is transformed by the Lorentz transformation[4].

Proof

Let a Lorentz transformation of the moving coordinate (x-direction) be

$$
\begin{aligned}
ct' &= \gamma(ct-\beta x)\\
x' &= \gamma(x-\beta ct)\\
y' &= y\\
z' &= z
\end{aligned}
$$

Then the transformation of $dV \wedge dct$ is

$$dV' \wedge dct' = dx' \wedge dy \wedge dz \wedge dct' = \gamma(dx-\beta dct)\wedge dy \wedge dz \wedge \gamma(dct-\beta dx) = \gamma^2(1-\beta^2)\, dx \wedge dy \wedge dz \wedge dct = dV \wedge dct,$$

and the transformation of $dS=(dV,\ d\mathbb{S}\wedge dct)$ is

$$
\begin{aligned}
&dV' = dx' \wedge dy \wedge dz = \gamma(dx-\beta dct)\wedge dy \wedge dz = \gamma(dx\wedge dy\wedge dz - \beta dy \wedge dz \wedge dct) = \gamma(dV - \beta dS_x)\\
dS':\ &dS_{x'} = dy \wedge dz \wedge dct' = dy \wedge dz \wedge \gamma(dct-\beta dx) = \gamma(dy\wedge dz\wedge dct - \beta dx \wedge dy \wedge dz) = \gamma(dS_x - \beta dV)\\
&dS_{y'} = dz \wedge dx' \wedge dct' = dz \wedge \gamma(dx-\beta dct)\wedge \gamma(dct-\beta dx) = \gamma^2(1-\beta^2)\, dz \wedge dx \wedge dct = dS_y\\
&dS_{z'} = dx' \wedge dy \wedge dct' = \gamma(dx-\beta dct)\wedge dy \wedge \gamma(dct-\beta dx) = \gamma^2(1-\beta^2)\, dx \wedge dy \wedge dct = dS_z.
\end{aligned}
$$

We represent the integral equation by the four components then

$$\int_U \begin{pmatrix} \partial_t & \\ & \nabla \end{pmatrix} \begin{pmatrix} E_t & \\ & \mathbb{E} \end{pmatrix} dV \wedge dct = \int_{\partial U} \begin{pmatrix} dV & \\ & d\mathbb{S}\wedge dct \end{pmatrix} \begin{pmatrix} E_t & \\ & \mathbb{E} \end{pmatrix}$$

$$\therefore \int_U \begin{pmatrix} \partial_t E_t + \mathrm{div}\,\mathbb{E} \\ \mathrm{grad} E_t + \partial_t \mathbb{E} - i\ \mathrm{rot}\,\mathbb{E} \end{pmatrix} dV \wedge dct = \int_{\partial U} \begin{pmatrix} E_t dV + \mathbb{E}\cdot d\mathbb{S}\wedge dct \\ E_t d\mathbb{S}\wedge dct + \mathbb{E} dV + i\mathbb{E}\times d\mathbb{S}\wedge dct \end{pmatrix}.$$

This equation is the same as the Gauss' theorem (1), (2), (3) as follows :

Real part[7] is

$$\int_U (\partial_t E_t + \mathrm{div}\mathbb{E})\, dV \wedge dt = \int_{\partial U} E_t dV + \mathbb{E}\cdot d\mathbb{S}\wedge dt$$

, i. e.,

i) $\displaystyle\int_U \partial_t E_t \underline{dx \wedge dy \wedge dz} \wedge dt = \int_{\partial U} E_t \underline{dx \wedge dy \wedge dz},$

and

ii) $\displaystyle\int_U \mathrm{div}\mathbb{E}\, dx \wedge dy \wedge dz \wedge \underline{dt} = \int_{\partial U} \mathbb{E}\cdot (dy\wedge dz,\ dz \wedge dx,\ dx \wedge dy)\, \underline{dt}.$

This formula is the same one as Gauss' theorem (1).

Imaginary part[7] is

$$\int_U (\mathrm{grad}E_t + \partial_t \mathbb{E} - i\ \mathrm{rot}\mathbb{E})\, dV \wedge dt = \int_{\partial U} E_t d\mathbb{S}\wedge dt + \mathbb{E} dV + i\mathbb{E}\times d\mathbb{S}\wedge dt$$

, i. e.,

i) $\displaystyle\int_U \mathrm{grad}\ E_t dx \wedge dy \wedge dz \wedge \underline{dt} = \int_{\partial U} E_t (dy \wedge dz,\ dz \wedge dx,\ dx \wedge dy) \wedge \underline{dt}.$

This formula is the same one as Gauss' theorem (3).

ii) $\displaystyle\int_U \partial_t \mathbb{E}\, \underline{dx \wedge dy \wedge dz} \wedge dt = \int_{\partial U} \mathbb{E}\, \underline{dx \wedge dy \wedge dz},$

and

iii) $\int_U \mathrm{rot}\,\mathbb{E}dx\wedge dy\wedge dz\wedge \underline{dt} = \int_{\partial U} (dy\wedge dz,\ dz\wedge dx,\ dx\wedge dy)\wedge \underline{dt}\times \mathbb{E}.$

This formula is the same one as Gauss' theorem (2).

q. e. d.

Proposition 8. (4-dimensional Green' theorem)

A following formulas are hold

$$\mathrm{Tr}\int_{\partial U}[dS((\tilde{D}\tilde{A})B)]-\mathrm{Tr}\int_{\partial U}[dS((\tilde{D}\tilde{B})A)]$$
$$=\mathrm{Tr}\int_U[(\Box\tilde{A})B)]\ dV\wedge dct-\mathrm{Tr}\int_U[(\Box\tilde{B})A]\ dV\wedge dct. \qquad \cdots(11)$$

And

$$\mathrm{Tr}\int_{\partial U}[ds((\tilde{D}\tilde{A})A)]=-\mathrm{Tr}\int_U[(\Box\tilde{A})A)]\ dV\wedge dct+\mathrm{Tr}\int_U[\widetilde{(\tilde{D}\tilde{A})}(\tilde{D}\tilde{A})]\ dV\wedge dct. \qquad \cdots(12)$$

where $\Box=\nabla^2-\partial^2/\partial t^2$, $\nabla=\partial/\partial x\ i+\partial/\partial y\ j+\partial/\partial z\ k$.

Proof

We can calculate directly as follows[4]:

$$\mathrm{Tr}\ [D((\tilde{D}\tilde{A})B)]=\mathrm{Tr}\ [(D\tilde{D}\tilde{A})B)]+\mathrm{Tr}\ \widetilde{[(\tilde{D}\tilde{A})(\tilde{D}\tilde{B}))]}\ ,$$
$$\mathrm{Tr}\ [D((\tilde{D}\tilde{B})A)]=\mathrm{Tr}\ [(D\tilde{D}\tilde{B})A)]+\mathrm{Tr}\ [(\tilde{D}\tilde{B})(\tilde{D}\tilde{A}))]\ , \qquad \cdots(13)$$

and

$$\mathrm{Tr}\ [D((\tilde{D}\tilde{A})B)]-\mathrm{Tr}\ [D((\tilde{D}\tilde{B})A)]=\mathrm{Tr}\ [(\Box\tilde{A})B)]-\mathrm{Tr}\ [(\Box\tilde{B})A]\ . \qquad \cdots(14)$$

We integrate (14) on the domain U, and ∂U then

$$\mathrm{Tr}\int_{\partial U}[dS((\tilde{D}\tilde{A})B)]-\mathrm{Tr}\int_{\partial U}[dS((\tilde{D}\tilde{B})A)]$$
$$=\mathrm{Tr}\int_U[D((\tilde{D}\tilde{A})B)]\ dV\wedge dct-\mathrm{Tr}\int_U[D((\tilde{D}\tilde{B})A]\ dV\wedge dct$$
$$=\mathrm{Tr}\int_U[(\Box\tilde{A})B)]\ dV\wedge dct-\mathrm{Tr}\int_U[(\Box\tilde{B})A]\ dV\wedge dct.$$

Let B=A in (15) and $D\tilde{D}=-\Box E$ (E : unit matrix), then

$$\mathrm{Tr}\ [D((\tilde{D}\tilde{A})A)]=\mathrm{Tr}\ [(D\tilde{D}\tilde{A})A)]+\mathrm{Tr}\ \widetilde{[(\tilde{D}\tilde{A})}(\tilde{D}\tilde{A})]\ . \qquad \cdots(15)$$

And integrate these (15) on the domain U, and ∂U then

$$\mathrm{Tr}\int_{\partial U}dS((\tilde{D}\tilde{A})A)=\mathrm{Tr}\int_U D((\tilde{D}\tilde{A})A)\ dV\wedge dct$$
$$=-\mathrm{Tr}\int_U[(\Box\tilde{A})A)]\ dV\wedge dct+\mathrm{Tr}\int_U[\widetilde{(\tilde{D}\tilde{A})}(\tilde{D}\tilde{A})]\ dV\wedge dct.$$

q. e. d.

§3. d'Alembert equation

Proposition 9.

The following formula is holds for a potential ϕ(ct, x, y, z) in the 4-dimensional time space :

$$\int_{\partial(U_r-U_r')}(\partial_t\phi)/r\ dV-(\mathrm{grad}\ \phi)/r\cdot d\mathbb{S}+\phi\ \mathrm{grad}\ 1/r\cdot d\mathbb{S}=-\int_{U_r-U_r'}(\Box\phi)/r\ dV\wedge dct$$

where $dV=dx\wedge dy\wedge dz$, $d\mathbb{S}\wedge dct=(dy\wedge dz,\ dz\wedge dx,\ dx\wedge dy)\wedge dct$, $dV\wedge dct=dx\wedge dy\wedge dz\wedge dct$.

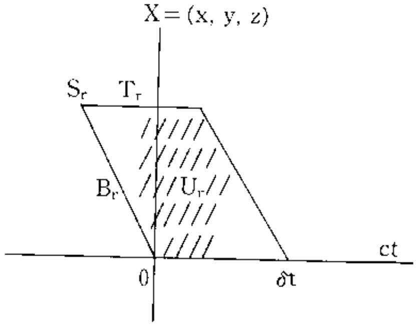
Proof

In an equation (14) (Prop. 8), let

$$A=\begin{pmatrix}\phi & \\ & \mathbb{A}\end{pmatrix} \quad \text{and} \quad B=\begin{pmatrix}1/r & \\ & 0\end{pmatrix}$$

then

$$\Box\tilde{B}=0$$



And when $r>r'>0$,

$S_r=\{(cT\quad, X)/|X|=r,\ cT=-r\}$: sphere

$T_r=\{(cT+ct, X)/|X|=r,\ cT=-r,\ 0\leq ct\leq \delta t\}$

$B_r=\{(cT\quad, X)/|X|=r,\ -r\leq cT\leq 0\}$: ball

$U_r=\{(cT+ct, X)/|X|=r,\ -r\leq cT\leq 0,\ 0\leq ct\leq \delta t\}$.

Therefore

$$\mathrm{Tr}\int_{\partial(U_r-U_{r'})} dS\cdot\left(\left(\begin{pmatrix}\partial_t & \\ & -\nabla\end{pmatrix}\begin{pmatrix}\phi & \\ & -\mathbb{A}\end{pmatrix}\right)\begin{pmatrix}1/r & \\ & 0\end{pmatrix}-\left(\begin{pmatrix}\partial_t & \\ & -\nabla\end{pmatrix}\begin{pmatrix}1/r & \\ & 0\end{pmatrix}\right)\begin{pmatrix}\phi & \\ & \mathbb{A}\end{pmatrix}\right)$$

$$=-\mathrm{Tr}\int_{U_r-U_{r'}}\left(\Box\begin{pmatrix}\phi & \\ & \mathbb{A}\end{pmatrix}\right)\begin{pmatrix}1/r & \\ & 0\end{pmatrix}dV\wedge dct.$$

$$\therefore\quad \mathrm{Tr}\int_{\partial(U_r-U_{r'})} dS\cdot\left(\begin{pmatrix}(\partial_t\phi)/r & \\ & -(\mathrm{grad}\phi)/r\end{pmatrix}-\begin{pmatrix}0 & \\ & -\mathrm{grad}\ 1/r\ \phi\end{pmatrix}\right)$$

$$+\mathrm{Tr}\int_{\partial(U_r-U_{r'})} dS\cdot\left(\begin{pmatrix}(\mathrm{div}\,\mathbb{A})/r & \\ & -(\partial_t\mathbb{A})/r-i(\mathrm{rot}\mathbb{A})/r\end{pmatrix}-\begin{pmatrix}-\mathrm{grad}\ 1/r\cdot\mathbb{A} & \\ & i\ \mathrm{grad}\ 1/r\times\mathbb{A}\end{pmatrix}\right)$$

$$=-\mathrm{Tr}\int_{U_r-U_{r'}}\left(\Box\begin{pmatrix}\phi & \\ & \mathbb{A}\end{pmatrix}\right)\cdot\begin{pmatrix}1/r & \\ & 0\end{pmatrix}dV\wedge dct$$

where $dS=\begin{pmatrix}dV & \\ & d\mathbb{S}\wedge dct\end{pmatrix}$

and

$dV=dx\wedge dy\wedge dz,\ d\mathbb{S}\wedge dct=(dy\wedge dz,\ dz\wedge dx,\ dx\wedge dy)\wedge dct,\ dV\wedge dct=dx\wedge dy\wedge dz\wedge dct.$

Therefore

$$\therefore\quad \int_{\partial(U_r-U_{r'})}(\partial_t\phi)/r\ dV-((\mathrm{grad}\ \phi)/r-\phi\ \mathrm{grad}(1/r))\cdot d\mathbb{S}\wedge dct$$

$$+\int_{\partial(U_r-U_{r'})}((\mathrm{div}A)/r+\mathbb{A}\cdot\mathrm{grad}(1/r))dV-(\partial_t\mathbb{A})/r\cdot d\mathbb{S}\wedge dct$$

$$-i\int_{\partial(U_r-U_{r'})}((\mathrm{rot}\mathbb{A})/r-\mathbb{A}\times\mathrm{grad}(1/r))\cdot d\mathbb{S}\wedge dct=-\int_{U_r-U_{r'}}(\Box\phi)/r\ dV\wedge dct$$

The 3rd and 4th term are

$$\int_{\partial(U_r-U_{r'})}((\mathrm{div}\mathbb{A})/r+\mathbb{A}\cdot\mathrm{grad}(1/r))dV-(\partial_t\mathbb{A})/r\cdot d\mathbb{S}\wedge dct$$

$$= \int_{\partial(U_r-U_{r'})} \mathrm{div}(\mathbb{A}/r)\,dV - (\partial_t\mathbb{A})/r\cdot d\mathbb{S}\wedge dct$$
$$=0.$$

The 5th term is

$$\int_{\partial(U_r-U_{r'})} ((\mathrm{rot}\mathbb{A})/r - \mathbb{A}\times \mathrm{grad}(1/r))\cdot d\mathbb{S}\wedge dct.$$
$$= \int_{\partial(U_r-U_{r'})} \mathrm{rot}(\mathbb{A}/r)\cdot d\mathbb{S}\wedge dct$$
$$= \int_{U_r-U_{r'}} \mathrm{div}\ \mathrm{rot}(\mathbb{A}/r)\,dV\wedge dct$$
$$=0.$$

Thetrefore

$$\therefore\quad \int_{\partial(U_r-U_{r'})} (\partial_t\phi)/r\ dV - (\mathrm{grad}\ \phi)/r\cdot d\mathbb{S}\wedge dct + \phi\ \mathrm{grad}\ 1/r\cdot d\mathbb{S}\wedge dct = -\int_{U_r-U_{r'}} (\square\phi)/r\ dV\wedge dct.$$

q. e. d.

Theorem 10.

Let $\begin{pmatrix}\phi\\ -\mathbb{A}\end{pmatrix}$ be a solution of the Laplace-Poisson's equation in 4-dimensional time-space , i. e.,

$$\square\begin{pmatrix}\phi\\ -\mathbb{A}\end{pmatrix} = -4\pi\begin{pmatrix}\rho\\ -\mathbb{J}\end{pmatrix},$$

then this solution satisfies the following formula

$$4\pi\begin{pmatrix}\phi\\ -\mathbb{A}\end{pmatrix}(0) = \int_{B_r}\begin{pmatrix}\rho\\ -\mathbb{J}\end{pmatrix}(-r,P)/r\,dv_P\quad, r=OP$$
$$+1/r^2\int_{S_r}\begin{pmatrix}\phi\\ -\mathbb{A}\end{pmatrix}(\mathbb{r}/r)\cdot d\mathbb{S}$$
$$+1/r\int_{S_r}\left[(d\mathbb{S}\cdot\mathrm{grad})\begin{pmatrix}\phi\\ -\mathbb{A}\end{pmatrix} + \partial_t\begin{pmatrix}\phi\\ -\mathbb{A}\end{pmatrix}(\mathbb{r}/r)\cdot d\mathbb{S}\right].$$

Proof

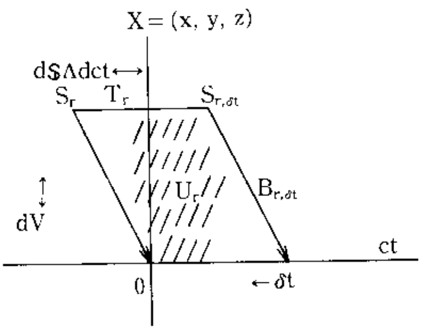
In proposition 9, when $r>r'>0$

$$\int_{\partial(U_r-U_{r'})} (\partial_t\phi)/r\ dV - (\mathrm{grad}\ \phi)/r\cdot d\mathbb{S}\wedge dct + \phi\ \mathrm{grad}\ 1/r\cdot d\mathbb{S}\wedge dct = -\int_{U_r-U_{r'}} (\square\phi)/r\ dV\wedge dct \quad\cdots (18)$$

where $dV = dx\wedge dy\wedge dz$, $d\mathbb{S}\wedge dct = (dy\wedge dz,\ dz\wedge dx,\ dx\wedge dy)\wedge dct$, $dV\wedge dct = dx\wedge dy\wedge dz\wedge dct$.

Let

$S_{r,\delta t} = \{(cT+\delta t, X)/|X|=r,\ cT=-r\}$
$S_r\ \ = \{(cT+ct, X)/|X|=r,\ cT=-r,\ 0\leqq ct\leqq \delta t\}$
$B_{r,\delta t} = \{(cT+\delta t, X)/|X|=r,\ -r\leqq cT\leqq 0\}$
$U_r\ \ = \{(cT+ct, X)/|X|=r,\ -r\leqq cT\leqq 0,\ 0\leqq ct\leqq \delta t\}$.



We divide the both side of (18) by δt and $\delta t\to 0$ then $U_r\to B_r$, and

$dV\wedge dct = dx\wedge dy\wedge dz\wedge dct \to dV = dx\wedge dy\wedge dz$ on B_r,
$T_r \to S_r$, and
$dS = d\mathbf{S}\wedge dct \to d\mathbf{S} = (dy\wedge dz,\ dz\wedge dx,\ dx\wedge dy)$ on S_r.

Therefore

$$\int_{B_{r,\delta t}} (\partial_t\phi)/r\ dV - \int_{B_r} (\partial_t\phi)/r\ dV \to \int_{B_r} (\partial_t^2\phi)/r\ dV,$$

$$\int_{B_{r,\delta t}} (\mathrm{grad}\phi)/r dV - \int_{B_r} (\mathrm{grad}\phi)/r dV \to \int_{B_r} (\partial_t\ \mathrm{grad}\phi)/r\ dV,$$

$$\int_{B_{r,\delta t}} \phi\,\mathrm{grad}(1/r) dV - \int_{B_r} \phi\,\mathrm{grad}(1/r) dV \to \int_{B_r} (\partial_t\phi)\mathrm{grad}(1/r) dV,$$

Then

1st term is

$$\int_{\partial(U_r - U_{r'})} (\partial_t\phi)/r\ dV \to \int_{B_r - B_{r'}} (\partial_t^2\phi)/r\ dV$$

2nd term is

$$-\int_{\partial(U_r - U_{r'})} (\mathrm{grad}\ \phi)/r\cdot d\mathbf{S}\wedge dct \to -\int_{S_r} (\mathrm{grad}\ \phi)/r\cdot d\mathbf{S} - \int_{B_r - B_{r'}} (\partial_t\ \mathrm{grad}\ \phi)/r\cdot d\mathbf{S}\wedge dct + \int_{S_{r'}} (\mathrm{grad}\ \phi)/r\cdot d\mathbf{S}$$

3rd term is

$$\int_{\partial(U_r - U_{r'})} \phi\ \mathrm{grad}(1/r)\cdot d\mathbf{S}\wedge dct \to \int_{S_r} \phi\ \mathrm{grad}(1/r)\cdot d\mathbf{S} + \int_{B_r - B_{r'}} \partial_t\ \phi\ \mathrm{grad}(1/r)\cdot d\mathbf{S}\wedge dct - \int_{S_{r'}} \phi\ \mathrm{grad}(1/r)\cdot d\mathbf{S}$$

and right side term is

$$-\int_{U_r - U_{r'}} (\Box\phi)/r\ dV\wedge dct \to -\int_{B_r - B_{r'}} (\Box\phi)/r\ dV$$

Therefore

$$\therefore\ \underline{\int_{B_r - B_{r'}} (\partial_t^2\phi)/r\ dV}$$

$$-\int_{S_r} (\mathrm{grad}\ \phi)/r\cdot d\mathbf{S} - \underline{\int_{B_r - B_{r'}} (\partial_t\ \mathrm{grad}\phi)/r\cdot d\mathbf{S}\wedge dct} + \int_{S_{r'}} (\mathrm{grad}\ \phi)/r\cdot d\mathbf{S}$$

$$+\int_{S_r} \phi\ \mathrm{grad}(1/r)\cdot d\mathbf{S} + \underline{\int_{B_r - B_{r'}} \partial_t\phi\ \mathrm{grad}(1/r)\cdot d\mathbf{S}\wedge dct} - \int_{S_{r'}} \phi\ \mathrm{grad}(1/r)\cdot d\mathbf{S} = -\int_{B_r - B_{r'}} (\Box\phi)/r\ dV \qquad \cdots (19)$$

where $d\mathbf{S} = (dy\wedge dz,\ dz\wedge dx,\ dx\wedge dy)$, $dV = dx\wedge dy\wedge dz$

We transform the above underlined part as follows :

On the light corn $B_r - B_{r'}$,

$$\begin{aligned} d\mathbf{S}\wedge dct &= (dy\wedge dz,\ dz\wedge dx,\ dx\wedge dy)\wedge dct, \quad ct = r = (x^2+y^2+z^2)^{1/2} \\ &= (dy\wedge dz\wedge(xdx)/r,\ dz\wedge dx\wedge(ydy)/r,\ dx\wedge dy\wedge(zdz)/r) \\ &= (x,\ y,\ z)/r\ dx\wedge dy\wedge dz \\ &= \mathbf{r}/r\ dV \end{aligned}$$

then

$$\int_{B_r - B_{r'}} (\partial_t^2\phi)/r\ dV - \int_{B_r - B_{r'}} (\partial_t\ \mathrm{grad}\phi)/r\cdot d\mathbf{S}\wedge dct + \int_{B_r - B_{r'}} \partial_t\phi\ \mathrm{grad}(1/r)\cdot d\mathbf{S}\wedge dct$$

$$= \int_{B_r - B_{r'}} (\partial_t^2 \phi)\, \mathbb{r}/r \cdot \mathbb{r}/r^2 - \mathrm{grad}(\partial_t \phi) \cdot \mathbb{r}/r^2 - (\partial_t \phi)\, \mathrm{div}(\mathbb{r}/r^2)\, dV \quad (\because\ \mathrm{div}(\mathbb{r}/r^2) = -\mathrm{grad}(1/r) \cdot \mathbb{r}/r)$$

$$= -\int_{B_r - B_{r'}} \mathrm{div}_{ct=r}((\partial_t \phi)\, \mathbb{r}/r^2)\, dV \qquad (\because \phi = \phi(-r, x, y, z) \text{ on the light corn } B_r - B_{r'})$$

$$= -\int_{S_r} ((\partial_t \phi)\, \mathbb{r}/r^2) \cdot d\mathbb{S} + \int_{S_{r'}} ((\partial_t \phi)\, \mathbb{r}/r^2) \cdot d\mathbb{S}.$$

Therefore the formula (19) is

$$-\int_{S_{r'}} \phi\, \mathrm{grad}(1/r) \cdot d\mathbb{S} = -\int_{B_r - B_{r'}} (\Box \phi)/r\, dV - \int_{S_r} \phi\, \mathrm{grad}(1/r) \cdot d\mathbb{S}$$
$$+ \int_{S_r} [(\mathrm{grad}\ \phi)/r + (\partial_t \phi)\, \mathbb{r}/r^2] \cdot d\mathbb{S}$$
$$+ \int_{S_{r'}} [(\mathrm{grad}\ \phi)/r + (\partial_t \phi)\, \mathbb{r}/r^2] \cdot d\mathbb{S}.$$

When $r' \to 0$

the left side term is

$$-\int_{S_{r'}} \phi\, \mathrm{grad}(1/r) \cdot d\mathbb{S} = 1/r'^2 \int_{S_{r'}} \phi\ \mathbb{r}/r' \cdot d\mathbb{S} \to 4\pi \phi(0),$$

4th term in the right side is

$$\left| -\int_{S_{r'}} [(\mathrm{grad}\ \phi)/r + ((\partial_t \phi)\, \mathbb{r}/r^2)] \cdot d\mathbb{S} \right| = 1/r' \left| \int_{S_{r'}} [(\mathrm{grad}\ \phi + (\partial_t \phi)\, \mathbb{r}/r)] \cdot d\mathbb{S} \right|$$
$$= 4\pi r' \max_{U_{r'}} |\mathrm{grad}\ \phi \cdot \mathbb{r}/r + \partial_t \phi|$$
$$\to 0,$$

and $\Box \phi = -\rho$,

therefore

$$4\pi \phi(0) = \int_{B_r} \rho/r\, dV + 1/r^2 \int_{S_r} \phi(\mathbb{r}/r) \cdot d\mathbb{S} + 1/r \int_{S_r} [\mathrm{grad}\ \phi + (\partial_t \phi)\, \mathbb{r}/r] \cdot d\mathbb{S}. \qquad \cdots (16)$$

Especially at the point $\rho = 0$, then

$$4\pi \phi(0) = 1/r^2 \int_{S_r} (\phi\, \mathbb{r}/r) \cdot d\mathbb{S} - 1/r \int_{B_r} (\partial_t \phi)\, \mathrm{div}(\mathbb{r}/r)\, dV.$$

Because

$$\mathrm{div}_{ct=r}[\mathrm{grad}\ \phi + \partial_t \phi\, \mathbb{r}/r]$$
$$= [\mathrm{div\ grad}\ \phi - \partial_t \mathrm{grad}\ \phi \cdot \mathbb{r}/r] + [\mathrm{grad}(\partial_t \phi) \cdot \mathbb{r}/r - \partial_t^2 \phi + (\partial_t \phi)\, \mathrm{div}(\mathbb{r}/r)]$$
$$= \Box \phi + (\partial_t \phi)\, \mathrm{div}(\mathbb{r}/r)$$
$$= -\rho + (\partial_t \phi)\, \mathrm{div}(\mathbb{r}/r)$$

We apply this formula to each component of $\mathbb{A}$ then

$$4\pi \mathbb{A}(0) = \int_{B_r} \mathbb{J}/r\, dV + 1/r^2 \int_{S_r} \mathbb{A}(\mathbb{r}/r) \cdot d\mathbb{S} + 1/r \int_{S_r} [((\mathbb{r}/r \cdot \mathrm{grad})\mathbb{A} + (\partial_t \mathbb{A})](\mathbb{r}/r) \cdot d\mathbb{S}. \qquad \cdots (17)$$

Therefore

$$4\pi \begin{pmatrix} \phi \\ -\mathbb{A} \end{pmatrix} (0) = \int_{B_r} \begin{pmatrix} \rho \\ -\mathbb{J} \end{pmatrix} (-r, P)/r\, dv_P$$

$$+1/r^2\int_{S_r}\begin{pmatrix}\phi & \\ & -\mathbb{A}\end{pmatrix}(\mathbb{r}/r)\cdot d\mathbb{S}$$

$$+1/r\int_{S_r}\left[\left(\mathbb{r}/r\cdot \mathrm{grad}\right)\begin{pmatrix}\phi & \\ & -\mathbb{A}\end{pmatrix}+\partial_t\begin{pmatrix}\phi & \\ & -\mathbb{A}\end{pmatrix}\right](\mathbb{r}/r)\cdot d\mathbb{S}\,.$$

q. e. d.

§4. The existance of a potential

Proposition 11.

i) A matrix formula

$$\tilde{D}D\begin{pmatrix}\phi & \\ & -\mathbb{A}\end{pmatrix}=D\tilde{D}\begin{pmatrix}\phi & \\ & -\mathbb{A}\end{pmatrix}=-\Box\begin{pmatrix}\phi & \\ & -\mathbb{A}\end{pmatrix}$$

is equivalent to the following ones :

Real part is

$\Box\phi=\mathrm{div}\ \mathrm{grad}\ \phi-\partial^2\phi/\partial^2 t,$

$\partial/\partial t\ \mathrm{div}\mathbb{A}-\mathrm{div}\ \partial\mathbb{A}/\partial t=0,\ \ \mathrm{div}\ \mathrm{rot}\,\mathbb{A}=0.$

Imaginary part is

$\Box\mathbb{A}=\mathrm{grad}\ \mathrm{div}\mathbb{A}-\mathrm{rot}\ \mathrm{rot}\mathbb{A}-\partial^2\mathbb{A}/\partial^2 t,$

$\partial/\partial t\ \mathrm{grad}\phi-\partial/\partial t\ \mathrm{grad}\phi=0,\ \ \partial/\partial t\ \mathrm{rot}\mathbb{A}-\mathrm{rot}\ \partial\mathbb{A}/\partial t=0,\ \ \mathrm{rot}\ \mathrm{grad}\phi=0.$

ii)

$$\tilde{D}\Box\begin{pmatrix}\phi & \\ & -\mathbb{A}\end{pmatrix}=\Box\tilde{D}\begin{pmatrix}\phi & \\ & -\mathbb{A}\end{pmatrix}$$

is equivalent to the following ones :

Real part is

$\partial/\partial t(\Box\phi)=\Box(\partial/\partial t\ \phi),\ \ \mathrm{div}(\Box\mathbb{A})=\Box(\mathrm{div}\mathbb{A}).$

Imaginary part is

$\partial/\partial t(\Box\mathbb{A})=\Box(\partial/\partial t\mathbb{A}),\mathrm{grad}(\Box\phi)=\Box(\mathrm{grad}\ \phi),\mathrm{rot}(\Box\mathbb{A})=\Box(\mathrm{rot}\mathbb{A}).$

Proof

For i)

$$D\tilde{D}\begin{pmatrix}\phi & \\ & -\mathbb{A}\end{pmatrix}=D\begin{pmatrix}\partial\phi/\partial t+\mathrm{div}\,\mathbb{A} & \\ & -\mathrm{grad}\phi-\partial\mathbb{A}/\partial t-i\ \mathrm{rot}\,\mathbb{A}\end{pmatrix}$$

$$=\begin{pmatrix}\partial/\partial t(\partial\phi/\partial t+\mathrm{div}\mathbb{A})+\mathrm{div}(-\partial\mathbb{A}/\partial t-\mathrm{grad}\ \phi-i\ \mathrm{rot}\mathbb{A}) \\ \mathrm{grad}(\partial\phi/\partial t+\mathrm{div}\mathbb{A})+\partial/\partial t(-\partial\mathbb{A}/\partial t-\mathrm{grad}\phi-i\ \mathrm{rot}\mathbb{A}) \\ -i\ \mathrm{rot}(-\partial\mathbb{A}/\partial t-\mathrm{grad}\phi-i\ \mathrm{rot}\mathbb{A})\end{pmatrix}$$

$$=\begin{pmatrix}(\partial^2\phi/\partial^2 t-\mathrm{div}\ \mathrm{grad}\phi)+(\partial/\partial t\ \mathrm{div}\mathbb{A}-\mathrm{div}\ \partial\mathbb{A}/\partial t)-i(\mathrm{div}\ \mathrm{rot}\mathbb{A}) \\ -(\partial^2\mathbb{A}/\partial^2 t-\mathrm{grad}\ \mathrm{div}\mathbb{A}+\mathrm{rot}\ \mathrm{rot}\mathbb{A})-(\mathrm{grad}\ \partial\phi/\partial t-\partial/\partial t\ \mathrm{grad}\phi) \\ -i(\partial/\partial t\ \mathrm{rot}\mathbb{A}-\mathrm{rot}\ \partial\mathbb{A}/\partial t)-i(\mathrm{rot}\ \mathrm{grad}\phi)\end{pmatrix}$$

For ii)

$$\tilde{D}\Box\begin{pmatrix}\phi & \\ & -\mathbb{A}\end{pmatrix}=\begin{pmatrix}\partial_t\Box & \\ & -\nabla\Box\end{pmatrix}\begin{pmatrix}\phi & \\ & -\mathbb{A}\end{pmatrix}$$

$$=\begin{pmatrix}\partial/\partial t(\Box\phi)+\mathrm{div}(\Box\mathbb{A}) & \\ & -\partial/\partial t(\Box\mathbb{A})-\mathrm{grad}(\Box\phi)-i\ \mathrm{rot}(\Box\mathbb{A})\end{pmatrix},$$

and

$$\Box\tilde{D}\begin{pmatrix}\phi \\ -\mathbb{A}\end{pmatrix} = \begin{pmatrix}\sqcup(\partial\phi/\partial t)+\sqcap(\mathrm{div}\mathbb{A}) \\ -\Box(\partial\mathbb{A}/\partial t)-\Box(\mathrm{grad}\ \phi)-i\ \sqcap(\mathrm{rot}\mathbb{A})\end{pmatrix}.$$

q. e. d.

Proposition 12.

Let $\begin{pmatrix}\phi \\ -\mathbb{A}\end{pmatrix}$ be a charge and current on the space time then a retarded potential,

then

$$\begin{pmatrix}\phi \\ -\mathbb{A}\end{pmatrix}(Q) = \int_V \begin{pmatrix}\rho \\ -\mathbb{J}\end{pmatrix}(ct-r,\ P)/r\,dv_P,\ r=PQ$$

V : whole space

satisfies the Laplace-poisson's equation

$$\sqcup\begin{pmatrix}\phi \\ -\mathbb{A}\end{pmatrix} = -4\pi\begin{pmatrix}\rho \\ -\mathbb{J}\end{pmatrix}.$$

Proof

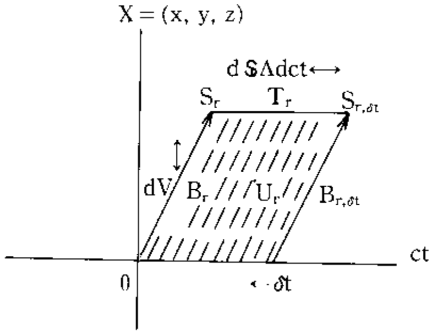
Let

$S_{r,\delta t} = \{(cT+\delta t,\ X)/|X|=r,\ cT=r\}$

$T_r\ = \{(cT+ct,\ X)/|X|=r,\ cT=r,\ 0\leqq ct\leqq \delta t\}$

$B_{r,\delta t} = \{(cT+\delta t,\ X)/|X|=r,\ 0\leqq cT\leqq r\}$

$U_r\ = \{(cT+ct,\ X)/|X|=r,\ 0\leqq cT\leqq r,\ 0\leqq ct\leqq \delta t\}$



Let ρ, $\mathbb{A}$ be stationary charge and current

and r' is arbitrary then

$$\int_{U_{r'}}\Box\begin{pmatrix}\phi \\ -\mathbb{A}\end{pmatrix} dV'\Lambda dct'$$

$$=\int_{U_{r'}}\left[\Box\int_{V\cap B_{r'}}\begin{pmatrix}\rho \\ -\mathbb{J}\end{pmatrix}/r\ dV+\sqcap\int_{V\setminus B_{r'}}\begin{pmatrix}\rho \\ -\mathbb{J}\end{pmatrix}/r\ dV\right] dV'\Lambda dct'$$

$$=-\int_{U_{r'}}(D(\int_{B_{r'}}\tilde{D}(\begin{pmatrix}\rho \\ -\mathbb{J}\end{pmatrix}/r)dV)dV'\Lambda dct' \qquad \because \sqcup\int_{V\setminus B_{r'}}\begin{pmatrix}\rho \\ -\mathbb{J}\end{pmatrix}(ct-r,\ p)/r\ dV_p=0$$

$$=-\int_{\partial U_{r'}}dS'(\int_{B_{r'}}\tilde{D}\left(\begin{pmatrix}\rho \\ -\mathbb{J}\end{pmatrix}/r\right)dV'$$

$$=-\int_{B_{r'}}(\int_{\partial U_{r'}}dS'\tilde{D}\left(\begin{pmatrix}\rho \\ -\mathbb{J}\end{pmatrix}/r\right))dV'$$

$$=-\int_{B_{r'}}(\int_{\partial U_{r'}}dS'\tilde{D}\left(\begin{pmatrix}\rho \\ -\mathbb{J}\end{pmatrix}/r\right))dV'$$

$$=-\int_{B_{r'}}\left(\int_{\partial U_{r'}}\begin{pmatrix}dV' \\ d\mathbb{S}'\Lambda dct'\end{pmatrix}\begin{pmatrix}\partial_t(\rho/r)+\mathrm{div}(\mathbb{J}/r) \\ -\mathrm{grad}(\rho/r)-\partial_t(\mathbb{J}/r)-i\ \mathrm{rot}(\mathbb{J}/r)\end{pmatrix}\right)dV'$$

$$=-\int_{B_{r'}}\left(\int_{\partial U_{r'}}\begin{pmatrix}dV'(\partial_t(\rho/r)+\mathrm{div}(\mathbb{J}/r))+d\mathbb{S}'\Lambda dct'(-\mathrm{grad}(\rho/r)-\partial_t(\mathbb{J}/r)-i\ \mathrm{rot}(\mathbb{J}/r)) \\ d\mathbb{S}'\Lambda dct'(\partial_t(\rho/r)+\mathrm{div}(\mathbb{J}/r))+dV'(-\mathrm{grad}(\rho/r)-\partial_t(\mathbb{J}/r)-i\ \mathrm{rot}(\mathbb{J}/r)) \\ -id\mathbb{S}'\Lambda dct'\times(-\mathrm{grad}(\rho/r)-\partial_t(\mathbb{J}/r)-i\ \mathrm{rot}(\mathbb{J}/r))\end{pmatrix}\right)dV'$$

When $\delta t \to 0$, then

$$\int_{B_{r'}} \square \begin{pmatrix} \phi \\ -\mathbb{A} \end{pmatrix} dV'$$

$$= -\int_{B_{r'}} \left(\int_{S_{r'}} \begin{pmatrix} d\mathbb{S}'(-\mathrm{grad}(\rho/r) - i\,\mathrm{rot}(\mathbb{J}/r)) \\ d\mathbb{S}'(\mathrm{div}(\mathbb{J}/r)) \\ -id\mathbb{S}' \times (-\mathrm{grad}(\rho/r) - i\,\mathrm{rot}(\mathbb{J}/r)) \end{pmatrix} \right) dV'$$

$$= -4\pi \int_{B_{r'}} \begin{pmatrix} \rho \\ -\mathbb{J} \end{pmatrix} dV'.$$

q. e. d.

Theorem 13.
An arbitrary 4-dimensional vector field $(E_t, \mathbb{E})$ in time-space is represented by the some 4-dimensional vector potential $(\phi, \mathbb{A})$ as follows :

$$E_t = \partial\phi/\partial t + \mathrm{div}\,\mathbb{A}$$
$$\mathbb{E} = -\mathrm{grad}\,\phi - \partial\mathbb{A}/\partial t - i\,\mathrm{rot}\,\mathbb{A}$$

Proof

The Laplace-Poisson equation

$$\square \begin{pmatrix} E_t \\ \mathbb{E} \end{pmatrix}^{0} = -4\pi \begin{pmatrix} E_t \\ \mathbb{E} \end{pmatrix}$$

has retarded potential as a special solution, i. e.,

$$\begin{pmatrix} E_t \\ \mathbb{E} \end{pmatrix}^{0} = \int_{V_o} \begin{pmatrix} E_t \\ \mathbb{E} \end{pmatrix} /r\, dv.$$

V_o : whole space

We substitute this solution into Laplace-Poisson equation and

let $\begin{pmatrix} \chi_0 \\ \chi \end{pmatrix}$ is an arbitrary 4 dimensional vector such that $\square \begin{pmatrix} \chi_0 \\ \chi \end{pmatrix} = 0$.

then

$$\begin{pmatrix} E_t \\ \mathbb{E} \end{pmatrix} = -1/4\pi \cdot \square \int_{V_o} \begin{pmatrix} E_t \\ \mathbb{E} \end{pmatrix} /r\, dv + \square \begin{pmatrix} \chi_0 \\ \chi \end{pmatrix}$$

$$= 1/4\pi \cdot \tilde{D}D \int_{V_o} \begin{pmatrix} E_t \\ \mathbb{E} \end{pmatrix} /r\, dv - \tilde{D}D \begin{pmatrix} \chi_0 \\ \chi \end{pmatrix}$$

$$= \tilde{D}\left(1/4\pi \cdot D \int_{V_o} \begin{pmatrix} E_t \\ \mathbb{E} \end{pmatrix} /r\, dv - D \begin{pmatrix} \chi_0 \\ \chi \end{pmatrix}\right)$$

$$= \tilde{D} \begin{pmatrix} \widetilde{\phi} \\ \mathbb{A} \end{pmatrix}$$

, i. e.,

$$E_t = \mathrm{div}\,\mathbb{A} + \partial\phi/\partial t$$
$$\mathbb{E} = -\mathrm{grad}\,\phi - \partial\mathbb{A}/\partial t - i\,\mathrm{rot}\,\mathbb{A}.$$

And

$$\begin{pmatrix} \tilde{\phi} \\ \mathbb{A} \end{pmatrix} = -1/4\pi \cdot D \int_{V_0} \begin{pmatrix} E_t \\ \mathbb{E} \end{pmatrix} /r\, dv + D \begin{pmatrix} \chi_0 \\ \chi \end{pmatrix}$$

, i. e.,

$$\phi = -\partial/\partial t \int \frac{E_t}{4\pi r} dV - \mathrm{div} \int \frac{\mathbb{E}}{4\pi r} dV - \mathrm{div}\ \chi + \partial\chi_0/\partial t$$

$$= -\int \frac{\mathrm{div}\ \mathbb{E} + \partial E_t/\partial t}{4\pi r} dV + \mathrm{div}\ \chi + \partial\chi_0/\partial t,$$

$$\mathbb{A} = \mathrm{grad} \int \frac{E_t}{4\pi r} dV + \partial/\partial t \int \frac{\mathbb{E}}{4\pi r} dV - i\ \mathrm{rot} \int \frac{\mathbb{E}}{4\pi r} dV - \mathrm{grad}\ \chi_0 - \partial\chi/\partial t + i\ \mathrm{rot}\ \chi$$

$$= \int \frac{\mathrm{grad}\ E_t + \partial \mathbb{E}/\partial t - i\ \mathrm{rot}\ \mathbb{E}}{4\pi r} dV - \mathrm{grad}\ \chi_0 - \partial\chi/\partial t + i\ \mathrm{rot}\ \chi.$$

q. e. d.

Corollary 14. (Helmholtz' theorem in 4-dimensional time-space.)

i) $(E_t, \mathbb{E})$ is represented by only scalar potential ϕ as

$$E_t = \partial\phi/\partial t$$
$$\mathbb{E} = -\mathrm{grad}\ \phi$$

iff

$$\mathrm{ROT}(E_t, \mathbb{E}) = \mathrm{grad}\ E_t + \partial\mathbb{E}/\partial t - i\ \mathrm{rot}\ \mathbb{E} = 0.$$

ii) $(E_t, \mathbb{E})$ is represented by only vector potential $\mathbb{A}$ as

$$E_t = \mathrm{div}\ \mathbb{A}$$
$$\mathbb{E} = -\partial\mathbb{A}/\partial t - i\ \mathrm{rot}\ \mathbb{A}$$

iff

$$\mathrm{DIV}(E_t, \mathbb{E}) = \partial E_t/\partial t + \mathrm{div}\ \mathbb{E} = 0.$$

Corollary 15. (3-dimensional case)

An arbitrary 3-dimensional vector field $\mathbb{E} - i\mathbb{B}$ (i.e., $E_t = 0$) in time-space is represented by the some 4-dimensional real vector potential $(\phi, \mathbb{A})$ as follows :

$$\mathbb{E} - i\mathbb{B} = -\mathrm{grad}\ \phi - \partial\mathbb{A}/\partial t - i\ \mathrm{rot}\ \mathbb{A}.$$

〔Example 16.〕 (Potential of electromagnetic fields)

$$\begin{cases} \mathrm{rot}\ \mathbb{E} + \partial\mathbb{B}/\partial t = 0 & \cdots (8) \\ \mathrm{div}\ \mathbb{B} = 0 & \cdots (9) \\ \mathrm{div}\ \mathbb{E} = \rho & \cdots (10) \\ \mathrm{rot} B - \partial\mathbb{E}/\partial t = \mathbb{J} & \cdots (11). \end{cases}$$

We transform above equations as

$(8) \times i - (11)$ is

$$\partial(\mathbb{E} - i\mathbb{B})/\partial t - i\ \mathrm{rot}(\mathbb{E} - i\mathbb{B}) = -\mathbb{J}\ (\mathrm{real})$$

$-(9) \times i + (10)$ is

$$\mathrm{div}(\mathbb{E} - i\mathbb{B}) = \rho\ (\mathrm{real}).$$

Therefore,

there exist a 4-dimensional vector potential $(\phi, \mathbb{A})$ such that

$$\mathbb{E} - i\mathbb{B} = -\mathrm{grad}\ \phi - \partial\mathbb{A}/\partial t - i\ \mathrm{rot}\ \mathbb{A}$$

where

$$\phi = -\int \frac{\mathrm{div}\,\mathbb{E}}{4\pi r}\,dV = -\int \frac{\rho}{4\pi r}\,dV$$

$$\mathbb{A} = \int \frac{\partial E_t/\partial t - i\ \mathrm{rot}\ \mathbb{E}}{4\pi r}\,dV = -\int \frac{\mathbb{J}}{4\pi r}\,dV.$$

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