~~ボーア半径の内側~~

The inside of the Bohr radius

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Abstract

In this paper, we seek the orbit of electron which “inside” of the Bohr radius by solving the equation. There orbit is between r_1 (the Bohr radius) and $2R_0$ (the horizon of the electromagnetic force). The minimal radius orbit by the resonance is a Bohr orbit. However, by the “apsidal precession” resonance, there are some orbits inside of the Bohr radius in the nucleus.

1. Preliminaries

1.1 The relativistic Bohr radius and the horizontal orbit

In the classical, we get the Bohr radius r_1 by the balance in the orbit,

$$\frac{m_e v_1{}^2}{r_1}(the\ centrifubal\ force) = \frac{k_0 e^2}{r_1{}^2}(the\ central\ force)$$

$$\therefore\ m_e r_1 v_1{}^2 = k_0 e^2. \qquad (1)$$

And by the Planck constant i.e., angular moment $h = 2m_e \pi r_1 v_1$. (2)

Therefore, the Bohr radius is $r_1 = \dfrac{h^2}{(2\pi)^2 m_e k_0 e^2} = 5.2923\times 10^{-11}$.

But in “the relativistic” case, we use the relativistic angular moment as the Planck constant.

Then $m_e r_n \Big(\dfrac{v_n}{\sqrt{1-(\frac{v_n}{\mathrm{c}})^2}}\Big)^2 = \dfrac{k_0 e^2}{\mathrm{c}^2}\Big(\dfrac{\mathrm{d}\mathrm{c}t}{\mathrm{d}\tau}\Big)^2$ and $nh = 2m_e \pi r_n \dfrac{v_n}{\sqrt{1-(\frac{v_n}{\mathrm{c}})^2}}$ for $n = 1, 2, 3, \cdots$.

Therefore $(2\pi r_n \mathrm{c} m_e v_n)^2 = (nh)^2(\mathrm{c}^2 - v_n{}^2)$, $v_n{}^2 = \dfrac{k_0 e^2}{m_e r_n}$.

$$\therefore\ (2\pi r_n \mathrm{c} m_e)^2 \frac{k_0 e^2}{m_e r_n} = (nh)^2\Big(\mathrm{c}^2 - \frac{k_0 e^2}{m_e r_n}\Big). \qquad (3)$$

$$\therefore\ k_0 e^2 (2\pi \mathrm{c} m_e)^2 r_n{}^2 - (n\mathrm{c}h)^2 m_e r_n + k_0 e^2 (nh)^2 = 0. \qquad (4)$$

We solve the above equation, and then we get the orbital radius of electron

$$r_n = \frac{(nch)^2 m_e \pm \sqrt{\{(nch)^2 m_e\}^2 - 4k_0 e^2 (2\pi c m_e)^2 k_0 e^2 (n\mathrm{h})^2}}{2k_0 e^2 (2\pi c m_e)^2}$$

$$= \frac{(nh)^2 (1 + \sqrt{1 - (\frac{4\pi k_0 e^2}{n\mathrm{hc}})^2})}{2(2\pi)^2 k_0 e^2 m_e} \qquad (5)$$

(Cf. $r_n = \dfrac{(nh)^2}{(2\pi)^2 k_0 e^2 m_e}$ in classical).

Especially $n = 1$, we get $r_1 = \dfrac{h^2 (1 + \sqrt{1 - (\frac{4\pi k_0 e^2}{\mathrm{hc}})^2})}{2(2\pi)^2 k_0 e^2 m_e} = 5.29137856 \times 10^{-11}$. Moreover the formula (5) of r_n valid for $n < 1$ and indicated the limited radius in a circle orbit.

That is to say, when the case $(\dfrac{4\pi k_0 e^2}{nh\mathrm{c}})^2 = 1$, then $n_s = \dfrac{4\pi k_0 e^2}{h\mathrm{c}} = \dfrac{1}{68.5165}$, therefore

$r_{n_s} = \dfrac{(nh)^2}{2(2\pi)^2 k_0 e^2 m_e} = \dfrac{2k_0 e^2}{m_e \mathrm{c}^2} (= 2R_0) = 5.636 \times 10^{-15}$ is the limited radius as "the Sch warzschild radius" which says by gravity and in this point the speed is $v = \dfrac{\mathrm{c}}{\sqrt{2}}$.

We put the horizon of the electromagnetic force $R_0 = \dfrac{k_0 e^2}{m_e \mathrm{c}^2} = r_1 (\dfrac{v_1}{\mathrm{c}})^2$, then $\dfrac{R_0}{r_1} = (\dfrac{v_1}{\mathrm{c}})^2 = 0.0000532513$ is "the fine structure". And this point is minimal point of the

Energy function $F(r) (= m_e \mathrm{c} C_0) = \dfrac{m_e \mathrm{c}^2}{\sqrt{1 - \frac{k_0 e^2}{m_e \mathrm{c}^2 r}}} \mathrm{e}^{-\frac{k_0 e^2}{m_e \mathrm{c}^2 r}}$.

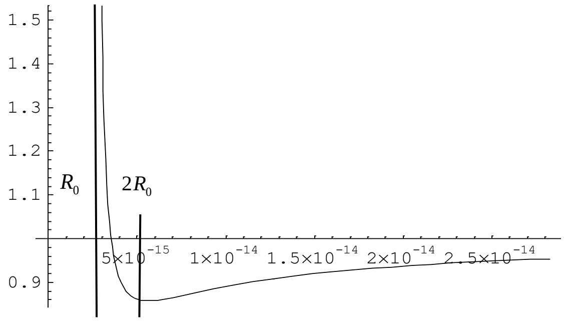

Fig. 1. The energy function.

1.2 The resonance of circle orbit

The Energy of which shift is the parabolic orbit into the circle orbit at any radius r,

$$\Delta E_{[kgm^2/s^2]} = m_e c^2 - \frac{m_e c^2}{\sqrt{1-(\frac{v}{c})^2}} e^{-\frac{R_0}{r}} \fallingdotseq \frac{1}{2} m_e v^2 = h_{s[kgm^2/s^2]} \nu^{s}{}_{[-]} (= h_{[kgm^2/s]} \nu_{[/s]}) . \qquad (6)$$

We considered the harmonic oscillator which frequent is ν and its energy is $\underline{\underline{h_{s[kgm^2/s^2]}}}$.

Then, the radius is $a = \sqrt{\frac{2h_s}{k}} = \sqrt{\frac{2h_s}{(2\pi\nu)^2 m_e}} = \frac{1}{2\pi\nu}\sqrt{\frac{2h_s}{m_e}} = \frac{1}{\omega}\sqrt{\frac{2h_s}{m_e}}$.

And the speed is $v_0(\text{constant}) = a\omega = \sqrt{\frac{2h_s}{m_e}} (= 0.0381416_{[m/s]})$.

And we put the surrounding frequency $\bar{\nu} = \frac{\nu}{2\pi(2r)} = \frac{\frac{1}{2}(2r)\nu}{\pi(2r)^2} (= \frac{Speed\ of\ area}{Area\ of\ circle})$, $\bar{\nu}_1 = \nu_1$.

where, ν_1 is the frequency of radiation light of the Bohr radius.

$$\text{Then}\quad \Delta E = h_s \nu^s = \frac{1}{2} m_e v_0{}^2 \cdot \nu^s \qquad (h_{s[kgm^2/s^2]} = \frac{1}{2} m_e v_0{}^2)$$

$$= \frac{1}{2} m_e v^2 = \frac{1}{2} m_e (\underbrace{2\pi(2r)\bar{\nu}}_{v})^2 \qquad (v = 2\pi(2r)\bar{\nu})$$

$$= \frac{1}{2} m_e (\underbrace{2\pi(2r_1)\bar{\nu}_1}_{v_1})^2 \times \left(\frac{\nu}{\nu_1}\right)^2 = 2\pi m_e r_1 v_1 \cdot \nu_1 \times \left(\frac{\nu}{\nu_1}\right)^2 = h \cdot \nu_1 \times \left(\frac{\nu}{\nu_1}\right)^2 \qquad (\bar{\nu}_1 = \nu_1).$$

(Example)

(i) When $x = \frac{\nu_1}{\nu} = 1$, $\nu_1 = 3.2898 \times 10^{15}{}_{[/s]}$,

$$\Delta E_1 \fallingdotseq \frac{1}{2} m v_1{}^2 = \frac{1}{2} m v_1 (2\pi(2r_1)\nu_1 \cdot 1) = \underbrace{\underline{\underline{2\pi m r_1 v_1}}}_{h} \cdot \nu_1 . \qquad (7)$$

(ii) When $x = \frac{\nu_1}{\nu} = 2$, $\nu_2 = \frac{\nu_1}{2^2} = 3.28984 \times 10^{15}{}_{[/s]} \times \frac{1}{4} = 8.2246 \times 10^{14}{}_{[/s]}$,

$$(\Delta E_1 \fallingdotseq) \frac{1}{2} m v_2{}^2 = \frac{1}{2} m v_2 (2\pi(2r_2)\nu_2 \cdot \frac{1}{2}) = 2\pi m r_2 v_2 \cdot \frac{1}{2} \nu_2 = \underbrace{\underline{\underline{2\pi m r_1 v_1}}}_{h} \nu_2 . \qquad (8)$$

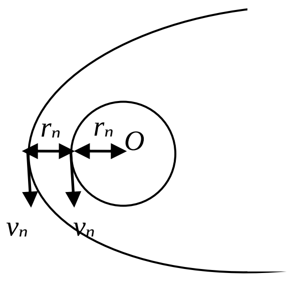

Fig. 2. The specific elliptic orbit.

(n) When $x = \frac{\nu_1}{\nu} = n$, $\nu_n = \frac{\nu_1}{n^2} = 3.28984 \times 10^{15}{}_{[/s]} \times \frac{1}{n^2}$,

$$(\Delta E_n \fallingdotseq) \frac{1}{2} m v_n{}^2 = \frac{1}{2} m v_n (2\pi(2r_n)\nu_n \cdot \frac{1}{n}) = 2\pi m r_n v_n \cdot \frac{1}{n} \nu_n = \underbrace{\underline{\underline{2\pi m r_1 v_1}}}_{h} \nu_n .$$

(9)

Moreover this system valid for the opposite direction $x = n = 1, \frac{1}{2}, \frac{1}{3}, \cdots$ and indicated the radius in a circle orbit. We put $x = n = \frac{1}{68}$ by the fine structure, $n_s = \frac{4\pi k_0 e^2}{\mathrm{hc}} = \frac{1}{68.5165}$.

Then $r_n = 6.42435 \times 10^{-15}$.

2. The Orbit Equation

The metric is $\mathrm{d}s^2 = -\mathrm{d}ct^2 + \mathrm{d}r^2 + r^2(\sin^2\theta \mathrm{d}\varphi^2 + d\theta^2)$.

We consider the two-body problem concerned with the nuclear and the electron as in one hydrogen atom. It is assumed that the electron moves on fixed surface. Therefore, we put $\theta = \frac{\pi}{2} - \mathrm{i}\Omega$. Ω is a parameter that relates to the angle of rotation on the orbit.

Then the metric is

$$\mathrm{d}s^2(= -\mathrm{d}c\tau^2) = -\mathrm{d}ct^2 + \mathrm{d}r^2 + r^2(\cosh^2\Omega \mathrm{d}\varphi^2 - d\Omega^2)\ (<0) \tag{10}$$

and the polar coordinate is (t, r, Ω, φ) ,

Then we get the equation of Kepler's type.

$$\begin{cases} (a)\ m_e \mathrm{c}\dfrac{\mathrm{d}ct}{\mathrm{d}\tau} = m_e \mathrm{c} C_0 e^{\frac{keQ}{m_e c^2 r}} \cdots \text{(the conservation of energy)} \\ (b)\ \dfrac{\mathrm{d}^2}{\mathrm{d}\tau^2}(r\sinh\Omega) = -\left(\dfrac{keQ}{m_e c^2 r^2}\dfrac{\mathrm{d}ct}{\mathrm{d}\tau}\right)\left(\tanh\Omega - r\cosh\Omega\dfrac{\mathrm{d}\varphi}{\mathrm{d}ct}\right)\cosh\Omega\left(\dfrac{\mathrm{d}ct}{\mathrm{d}\tau}\right) \\ \qquad\qquad \cdots \text{(the structure of space)} \\ (c)\ r^2\left\{\left(r\cosh\Omega\dfrac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)^2 - \left(r\dfrac{\mathrm{d}\Omega}{\mathrm{d}\tau}\right)^2\right\} = C^2 \cdots \text{(the law of equal areas)} \\ (d)\ r^2\cosh\Omega\dfrac{\mathrm{d}\varphi}{\mathrm{d}\tau} = C\cosh\Theta(\geqq 0), \quad r\dfrac{d\Omega}{d\tau} = -C\sinh\Theta \\ \quad \Theta = -\displaystyle\int\left(\dfrac{keQ}{m_e c^2 r^2}\dfrac{\mathrm{d}ct}{\mathrm{d}\tau} - \sinh\Omega\dfrac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)\mathrm{d}\tau \cdots\cdots \text{(the internal rotation)} \end{cases}$$

$$\begin{cases} ct = ct \\ x = r\cosh\Omega\cos\varphi \\ y = r\cosh\Omega\sin\varphi \\ iz = \mathrm{i}r\sinh\Omega \end{cases} .$$

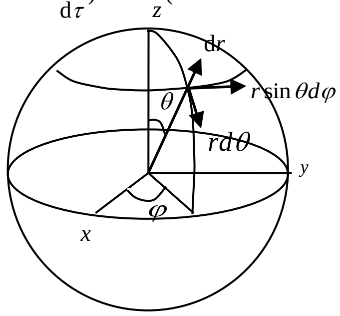

Fig. 3. The sphere type.

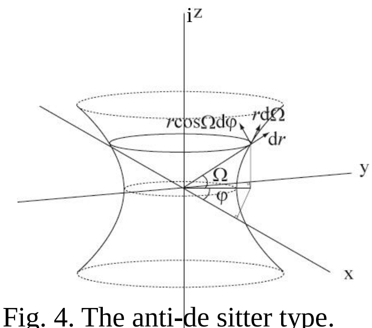

Fig. 4. The anti-de sitter type.

Where C_0 is the speed constant for the energy function and C is the area speed constant. The main equation is

$$(\frac{\mathrm{d}r}{\mathrm{dc}\tau})^2=(\frac{\mathrm{dc}t}{\mathrm{dc}\tau})^2-r^2((\cosh\Omega\frac{\mathrm{d}\varphi}{\mathrm{dc}\tau})^2-(\frac{\mathrm{d}\Omega}{\mathrm{dc}\tau})^2)-1$$

$$=(\frac{C_0}{\mathrm{c}}e^{\frac{R_0}{r}})^2-(\frac{C}{\mathrm{c}r})^2-1,\quad R_0=\frac{keQ}{m_e\mathrm{c}^2}\doteqdot 2.81795\times10^{-15}\mathrm{m}. \tag{11}$$

And we get the relation from the metric.

$$\frac{\mathrm{d}t}{\mathrm{d}\tau}=\frac{1}{\sqrt{1-(\frac{\mathrm{d}r}{\mathrm{dc}t})^2-(r\cosh\Omega\frac{\mathrm{d}\varphi}{\mathrm{dc}t})^2+(r\frac{\mathrm{d}\Omega}{\mathrm{dc}t})^2}}=\frac{1}{\sqrt{1-(\frac{\mathrm{d}r}{\mathrm{dc}t})^2}\sqrt{1-(r\cosh\Omega\frac{\mathrm{d}\varphi}{\mathrm{dc}t})^2}}.$$

By the relation $\frac{\mathrm{d}r}{\mathrm{dc}t}\cdot r\cosh\Omega\frac{\mathrm{d}\varphi}{\mathrm{dc}t}=-r\frac{\mathrm{d}\Omega}{\mathrm{dc}t}\ \therefore\ \frac{\mathrm{d}r}{\mathrm{dc}t}=-\frac{\mathrm{d}\Omega}{\cosh\Omega\mathrm{d}\varphi}$.

(The angular velocity)

When the curve $r=r(t)$ is given, $\frac{\mathrm{d}r}{\mathrm{dc}t}$ is calculated

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\tau}(=\frac{C}{r^2})=\sqrt{(\cosh\Omega\frac{\mathrm{d}\varphi}{\mathrm{d}\tau})^2-(\frac{\mathrm{d}\Omega}{\mathrm{d}\tau})^2}=\cosh\Omega\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\sqrt{1-(\underbrace{\frac{\mathrm{d}r}{\mathrm{dct}}}_{\text{"r" is given}})^2}. \tag{12}$$

Then $\frac{\mathrm{d}r}{\mathrm{d}\Phi}=-r\sqrt{-1+r^2\frac{C_0{}^2}{C^2}\mathrm{e}^{\frac{2R_0}{r}}-r^2\frac{\mathrm{c}^2}{C^2}}$ is a “orbit equation”.

And the orbit speed

$$r\frac{\mathrm{d}\Phi}{\mathrm{d}\tau}(=\frac{C}{r})=\sqrt{(r\cosh\Omega\frac{\mathrm{d}\varphi}{\mathrm{d}\tau})^2-(r\frac{\mathrm{d}\Omega}{\mathrm{d}\tau})^2}=r\cosh\Omega\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\sqrt{1-(\frac{\mathrm{d}r}{\mathrm{dc}t})^2},$$

$$\therefore r\cosh\Omega\frac{\mathrm{d}\varphi}{\mathrm{dc}\tau}=\frac{\frac{C}{\mathrm{c}r}}{\sqrt{1-(\frac{\mathrm{d}r}{\mathrm{dc}t})^2}}. \tag{13}$$

And more $\frac{r\cosh\Omega\frac{\mathrm{d}\varphi}{\mathrm{dc}t}}{\sqrt{1-(\frac{\mathrm{d}r}{\mathrm{dc}t})^2}\sqrt{1-(r\cosh\Omega\frac{\mathrm{d}\varphi}{\mathrm{dc}t})^2}}=r\cosh\Omega\frac{\mathrm{d}\varphi}{\mathrm{dc}\tau}=\frac{\frac{C}{\mathrm{c}r}}{\sqrt{1-(\frac{\mathrm{d}r}{\mathrm{dc}t})^2}}$.

Therefore,

$$r\cosh\Omega\frac{\mathrm{d}\varphi}{\mathrm{dc}t}=\frac{\frac{C}{\mathrm{c}r}}{\sqrt{1+(\frac{C}{\mathrm{c}r})^2}}\ \therefore\sqrt{1-(r\cosh\Omega\frac{\mathrm{d}\varphi}{\mathrm{dc}t})^2}=\frac{1}{\sqrt{1+(\frac{C}{\mathrm{c}r})^2}}. \tag{14}$$

(The speed constant for energy function and the speed constant)

$$\frac{\mathrm{dt}}{\mathrm{d}\tau}=\frac{1}{\sqrt{1-(\frac{\mathrm{d}r}{\mathrm{dc}t})^2}\sqrt{1-(r\cosh\Omega\frac{\mathrm{d}\varphi}{\mathrm{dc}t})^2}}=\frac{\sqrt{1+(\frac{C}{\mathrm{c}r})^2}}{\sqrt{1-(\frac{\mathrm{d}r}{\mathrm{dc}t})^2}}, \tag{15}$$

$$\frac{C_0}{\mathrm{c}}=\frac{\mathrm{dt}}{\mathrm{d}\tau}\mathrm{e}^{-\frac{R_0}{r}}=\frac{1}{\sqrt{1-(\frac{\mathrm{d}r}{\mathrm{dc}t})^2}\sqrt{1-(r\cosh\Omega\frac{\mathrm{d}\varphi}{\mathrm{dc}t})^2}}\mathrm{e}^{-\frac{R_0}{r}}$$

$$=\frac{1}{\sqrt{1-(\frac{\mathrm{d}r}{\mathrm{dc}t})^2}\sqrt{1-(\frac{\frac{C}{\mathrm{c}r}}{\sqrt{1+(\frac{C}{\mathrm{c}r})^2}})^2}}\mathrm{e}^{-\frac{R_0}{r}}=\frac{\sqrt{1+(\frac{C}{\mathrm{c}r})^2}}{\sqrt{1-(\frac{\mathrm{d}r}{\mathrm{dc}t})^2}}\mathrm{e}^{-\frac{R_0}{r}}. \tag{16}$$

Therefore

$$(\frac{C_0}{\mathrm{c}})^2=\frac{1+(\frac{C}{\mathrm{c}r})^2}{1-(\frac{\mathrm{d}r}{\mathrm{dc}t})^2}\mathrm{e}^{-2\frac{R_0}{r}}\ \therefore(\frac{C_0}{\frac{\mathrm{c}}{\sqrt{1-(\frac{\mathrm{d}r}{\mathrm{dc}t})^2}}\mathrm{e}^{-\frac{R_0}{r}}})^2-(\frac{C}{\mathrm{c}r})^2=1.$$

(Example) The circle orbit

When $\Omega\equiv c_0$ (Constant) $\therefore\frac{d\Omega}{d\tau}=0\ \therefore\frac{\mathrm{d}r}{\mathrm{dc}t}\equiv 0$ (Circle)

Then the equation of Kepler's type is

$$\begin{cases}(a')\ m_e\mathrm{c}\frac{\mathrm{dc}t}{\mathrm{d}\tau}=m_e\mathrm{c}C_0\mathrm{e}^{\frac{keQ}{m_e\mathrm{c}^2r}}\cdots\text{(the conservation of energy)}\\ (b')\ 0=-(\frac{keQ}{m_e\mathrm{c}^2r^2}\frac{\mathrm{dc}t}{\mathrm{d}\tau})(\tanh\Omega-r\cosh\Omega\frac{\mathrm{d}\varphi}{\mathrm{dc}t})\cosh\Omega(\frac{\mathrm{dc}t}{\mathrm{d}\tau})\cdots\ \text{(the structure of space)}\\ (c')\ r^2(r\cosh\Omega\frac{\mathrm{d}\varphi}{\mathrm{d}\tau})^2=C^2\cdots\text{(the law of equal areas)}\\ (d')\ r^2\cosh\Omega\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}=C(\geqq 0),\ 0=-C\sinh\Theta\\ \quad 0=-\int(\frac{keQ}{m_e\mathrm{c}^2r^2}\frac{\mathrm{dc}t}{\mathrm{d}\tau}-\sinh\Omega\frac{\mathrm{d}\varphi}{\mathrm{d}\tau})\mathrm{d}\tau\cdots\cdots\text{(the internal rotation)}\end{cases}$$

Where C_0 is the speed constant for energy function and C is the area speed constant.

The main equation is

$$(\frac{\mathrm{d}r}{\mathrm{dc}\tau})^2=(\frac{\mathrm{dc}t}{\mathrm{dc}\tau})^2-r^2(\cosh\Omega\frac{\mathrm{d}\varphi}{\mathrm{dc}\tau})^2-1$$

$$=(\frac{C_0}{\mathrm{c}}e^{\frac{R_0}{r}})^2-(\frac{C}{cr})^2-1,\quad R_0=\frac{keQ}{m_e\mathrm{c}^2}\fallingdotseq 2.81795\times 10^{-15}\mathrm{m}.$$

Then we get the relation

$$\tanh\Omega \equiv r\cosh\Omega\frac{\mathrm{d}\varphi}{\mathrm{d}ct} \tag{17}$$

in the circle and $\sinh\Omega\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}=\frac{keQ}{m_e c^2 r^2}\frac{\mathrm{d}ct}{\mathrm{d}\tau}$ by the equation (b') ,(d') of Kepler's type.

Then,

$$\tanh\Omega\cdot r\cosh\Omega\frac{\mathrm{d}\varphi}{\mathrm{d}ct}=r\sinh\Omega\frac{\mathrm{d}\varphi}{\mathrm{d}ct}=\frac{keQ}{m_e cr}=\frac{R_0}{r}. \tag{18}$$

Therefore, we get the relation: $\tanh\Omega=\sqrt{\frac{R_0}{r}}$.

Then, $\cosh\Omega=\frac{1}{\sqrt{1-\tanh^2\Omega}}=\frac{1}{\sqrt{1-\frac{R_0}{r}}}\ \therefore \sinh\Omega=\tanh\Omega\cosh\Omega=\frac{\sqrt{\frac{R_0}{r}}}{\sqrt{1-\frac{R_0}{r}}}$,

<u>* This means that the circle orbit which has no rotation i.e. $\Omega\equiv 0$ is not exist.</u>

$$\frac{\mathrm{dt}}{\mathrm{d}\tau}=\frac{1}{\sqrt{1-(r\cosh\Omega\frac{\mathrm{d}\varphi}{\mathrm{d}ct})^2}}=\frac{1}{\sqrt{1-\frac{R_0}{r}}}(=\cosh\Omega). \tag{19}$$

Especially for $r=2R_0$,

$$\tanh\Omega=\sqrt{\frac{R_0}{r}}=\sqrt{\frac{R_0}{2R_0}}=\frac{1}{\sqrt{2}},\quad \cosh\Omega=\frac{1}{\sqrt{1-\frac{R_0}{r}}}=\frac{1}{\sqrt{1-\frac{R_0}{2R_0}}}=\sqrt{2},$$

$$\sinh\Omega=\frac{\sqrt{\frac{R_0}{r}}}{\sqrt{1-\frac{R_0}{r}}}=\frac{\sqrt{\frac{R_0}{2R_0}}}{\sqrt{1-\frac{R_0}{2R_0}}}=1, \tag{20}$$

$$(\frac{C_0}{\mathrm{c}}\mathrm{e}^{\frac{R_0}{r}})^2-(\frac{C}{\mathrm{c}r})^2=1. \tag{21}$$

3. The flower orbit

The orbit equation for $\frac{1}{r}$ is

$$(\frac{\mathrm{d}\frac{1}{r}}{\mathrm{e}^{-\frac{R_0}{r}}\mathrm{d}\Phi})^2=(\frac{\mathrm{d}r}{r^2\mathrm{e}^{-\frac{R_0}{r}}\mathrm{d}\Phi})^2=(\frac{{C_0}^2}{C^2}\mathrm{e}^{\frac{2R_0}{r}}-\frac{1}{r^2}-\frac{\mathrm{c}^2}{C^2})\mathrm{e}^{\frac{2R_0}{r}},\mathrm{d}\theta=\mathrm{e}^{-\frac{R_0}{r}}\mathrm{d}\Phi.$$

(22)

And the orbit equation for r is

$$\frac{\mathrm{d}r}{\mathrm{e}^{-\frac{R_0}{r}}\mathrm{d}\Phi}=-r\sqrt{-1+r^2\frac{C_0^{\ 2}}{C^2}\mathrm{e}^{\frac{2R_0}{r}}-r^2\frac{\mathrm{c}^2}{C^2}\mathrm{e}^{\frac{R_0}{r}}},\mathrm{d}\theta=\mathrm{e}^{-\frac{R_0}{r}}\mathrm{d}\Phi. \tag{23}$$

We get the solution of this equation of inside of the Bohr radius.

(Example 1) The case $\alpha^2=1.25(=5/4)$

When the (classical) circle orbit (which is not exist) in the radius $r=2R_0(=5.63588\times 10^{-15})$

and $\Omega=0$, i.e., this is not consider Ω .

The speed constant foe energy function and the area speed constant are,

$$C=r\frac{\mathrm{c}\sqrt{\frac{R_0}{r}}}{\sqrt{1-\frac{R_0}{r}}}=2\mathrm{c}R_0\frac{\sqrt{\frac{R_0}{2R_0}}}{\sqrt{1-\frac{R_0}{2R_0}}}=2\mathrm{c}R_0=1.6895944905201704\times 10^{-6}\mathrm{m}^2/\mathrm{s},$$

$$C_0=\frac{\mathrm{c}}{\sqrt{1-\frac{R_0}{r}}}\mathrm{e}^{-\frac{R_0}{r}}=\frac{\mathrm{c}}{\sqrt{1-\frac{R_0}{2R_0}}}\mathrm{e}^{-\frac{R_0}{2R_0}}=\sqrt{2}\mathrm{c}\mathrm{e}^{-\frac{1}{2}}=2.5715114345599952\times 10^{8}\mathrm{m/s}.$$

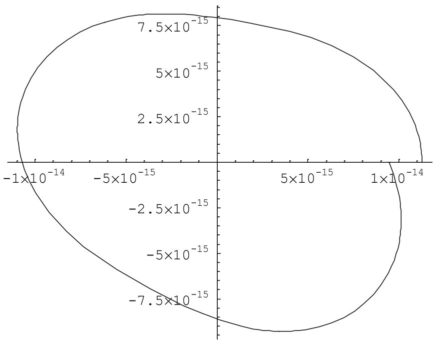

Fig. 5. Ex.1 for $0\leqq\theta\leqq 2\pi$.

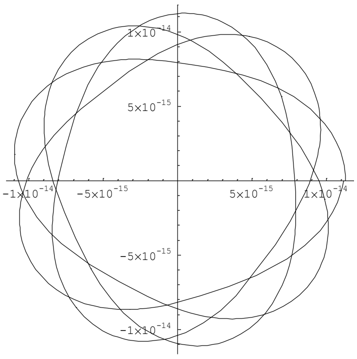

Fig. 6. Ex.1 for $0\leqq\theta\leqq 2\pi\times 4$.

Then the solution of the orbit equation is

```
r = -2.817940287410717`*^-15 +
                1.4089701437053578`*^-14
  ---------------------------------------------------
  1 + 0.5667500227486313` i Sin[1.118033988749895` θ]
```
. (24)

And the real part of this solution is

```
r = -2.817940287410717`*^-15 +
              1.4089701437053578`*^-14
    ------------------------------------------------
    1 + 0.3212055882855739` Sin²[1.118033988749895` θ]
```

. (25)

Therefore the coefficient of apsidal precession is $\alpha = 1.118033988749895$` ,

and $\alpha^2 = 1.25(=5/4)$.

When the apsidal precession is zero $\alpha = 0$ then the figure is as follows.

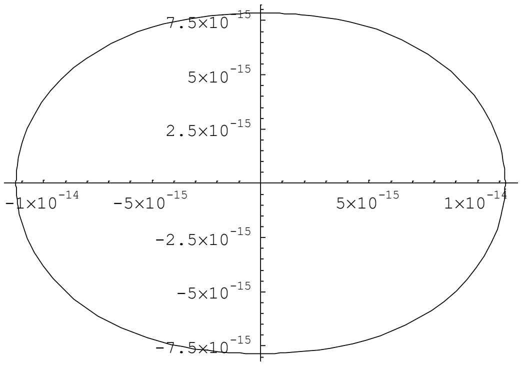

Fig. 7. Ex.1 for $\alpha = 0$.

(Example 2) The case $\alpha^2 = 1.2(=6/5)$

When the (classical) circle orbit in the radius $r = 1.01954 \times 10^{-14}$ which is the minimum radius of charge. The speed constant foe energy function and the area speed constant are

$C_0 = 2.67321 \times 10^8$m/s , $C = 1.88902 \times 10^{-6}$m^2/s .

Then the solution of the orbit equation is

```
r = -2.81794028741072`*^-15 +
              1.6907641709345915`*^-14
    ------------------------------------------------
    1 + 0.4789167960708896` i Sin[1.0954451151082842` θ]
```

. (26)

And the real part of this solution is

```
r = -2.81794028741072`*^-15 +
              1.6907641709345915`*^-14
    ------------------------------------------------
    1 + 0.22936129755880608` Sin²[1.0954451151082842` θ]
```

. (27)

Therefore the coefficient of apsidal precession is $\alpha = 1.0954451151082842$`,

and $\alpha^2 = 1.2(=6/5)$.

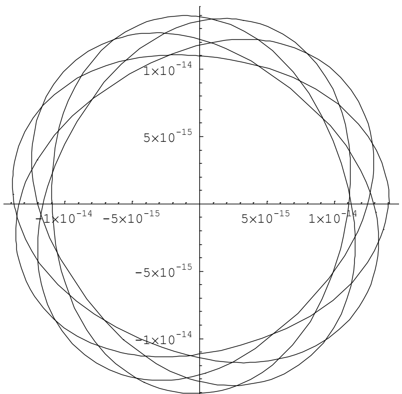

Fig. 8. Ex.2 for $0 \leqq \theta \leqq 2\pi \times 5$.

The case $\alpha^2 = 1.1666666666666666\cdot\cdot(=7/6)$ and $\alpha^2 = 1.1666666666666666\cdot\cdot(=7/6)$ are close to shape of the Example 2, and the next Example 3.

(Example 3) The case $\alpha^2 = 1.125(=9/8)$

When the (classical) circle orbit in the radius $r = 1.92421 \times 10^{-14}$ which is the minimum radius of charge. The speed constant foe energy function and the area speed constant are $C_0 = 2.802879 \times 10^8 \mathrm{m/s}$, $C = 2.389447 \times 10^{-6} \mathrm{m^2/s}$.

Then the solution of the orbit equation is

$$r = -2.8179402874107173`*^-15 + \frac{2.53614633104316`*^-14}{1 + 0.36467280089849763`\, i \,\mathrm{Sin}[1.0606601698880804`\,\theta]} . \qquad (28)$$

And the real part of this solution is

$$r = -2.8179402874107173`*^-15 + \frac{2.53614633104316`*^-14}{1 + 0.13298625171515513`\, \mathrm{Sin}^2[1.0606601698880804`\,\theta]} . \qquad (29)$$

Therefore the coefficient of apsidal precession is $\alpha = 1.0606601698880804`$,

and $\alpha^2 = 1.125(=9/8)$.

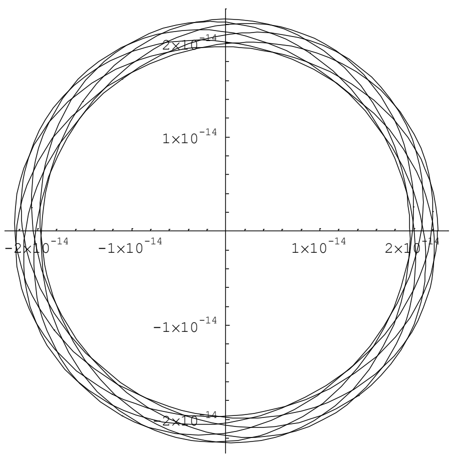

Fig. 9. Ex.3 for $0 \leqq \theta \leqq 2\pi \times 8$.

4. Conclusion

The electron orbit outside of the Bohr radius r_1 and $n^2 r_1 (n = 2, 3, \cdots)$ in the nucleus are by the resonance of the Shift Energy among the orbit.

And more the electron orbit inside of the Bohr radius in the nucleus become the flower orbit by the apsidal precision resonance of the orbit.

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