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〔論　　文〕

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Abstract

The Bohr atomic model is well-known for a hydrogen atom. The relation between an electron orbit and the light emitted from an atom was mostly solved by the Bohr atomic model. It is the purpose of this paper to show the Maxwell equation and an orbit for the equation of motion.

We derive the equation of motion of an electron by the modified Maxwell equation and try to improve the Bohr atomic model with the theory of relativity. We draw a better approximation as the result.

1. Introduction

Since the finding of "the matrix vector and the Lorentz form", the method has been applied to the Maxwell equation. We get the modified Maxwell equation including the time component. The authors extended the scope even to the "Electromagnetic and Gravitational Theory" and the "New Concept and Basic Tools" which treat the movement of planets and the Lorentz transformation. This time, an extended atomic model will be presented.

2. Preliminaries

2.1 The Maxwell equation by the Lorentz form

The existence of the time component E_t in the electromagnetic field $\boldsymbol{E}-\mathrm{i}c\boldsymbol{B}$, the 4-dimensional electromagnetic field for the derivative of the scalar potential ϕ and the vector potential $c\boldsymbol{A}$ are as follows:

$$
{}^{-}\begin{bmatrix} E_t & \\ & \boldsymbol{E}-\mathrm{i}c\boldsymbol{B} \end{bmatrix}^{+} = {}^{-}\begin{bmatrix} \dfrac{\partial}{\partial ct} & \\ & -\dfrac{\partial}{\partial \boldsymbol{r}} \end{bmatrix}^{-+}\begin{bmatrix} \phi & \\ & -c\boldsymbol{A} \end{bmatrix}^{+}
$$

$$
= {}^{-}\begin{bmatrix} \dfrac{\partial \phi}{\partial ct} + div\, c\boldsymbol{A} & \\ & -\dfrac{\partial c\boldsymbol{A}}{\partial ct} - grad\, \phi - \mathrm{i}\, rot\, c\boldsymbol{A} \end{bmatrix}^{+}.
$$

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Especially we consider the 4-dimensional potential $\phi(x, y, z) = -\frac{1}{4\pi\varepsilon}\frac{Q}{r}$ (ε is electric permittivity of the medium) and $\boldsymbol{A}(x, y, z) = 0$ which are caused by the stationary positive charge Q.

$${}^{-}\begin{bmatrix} E_t & \\ & \boldsymbol{E} - \mathrm{i}c\boldsymbol{B} \end{bmatrix}^{+} = {}^{-}\begin{bmatrix} \frac{\partial}{\partial \mathrm{c}t} & \\ & -\frac{\partial}{\partial \boldsymbol{r}} \end{bmatrix}^{-+}\begin{bmatrix} -\frac{1}{4\pi\varepsilon}\frac{Q}{r} & \\ & 0 \end{bmatrix}^{+} = \frac{1}{4\pi\varepsilon}{}^{-}\begin{bmatrix} 0 & \\ & \frac{\partial}{\partial \boldsymbol{r}}\left(\frac{Q}{r}\right) \end{bmatrix}^{+}.$$

Thus, the electric field is

$$\boldsymbol{E} = \frac{1}{4\pi\varepsilon}\frac{\partial}{\partial \boldsymbol{r}}\left(\frac{Q}{r}\right) = -\frac{Q}{4\pi\varepsilon r^2}\frac{\boldsymbol{r}}{r}.$$

Furthermore, the magnetic field and time component are

$\boldsymbol{B} = 0$ and $E_t = 0$.

In this case, the electric field of the 4-dimensional is the same result as using the 3-dimensional potential $V(x, y, z) = -\frac{1}{4\pi\varepsilon}\frac{Q}{r}$.

2.2 The 4-dimensional Coulomb - Lorentz force

The 4-dimensional electromagnetic field receives the 4-dimensional force on the charge density ρ and the current density stream $\boldsymbol{j}_s$ as follows[1], [2] :

$${}^{-}\begin{bmatrix} f_t & \\ & \boldsymbol{f} \end{bmatrix}^{-} = {}^{-}\begin{bmatrix} E_t & \\ & \boldsymbol{E} - \mathrm{i}c\boldsymbol{B} \end{bmatrix}^{+-}\begin{bmatrix} \rho & \\ & \frac{\boldsymbol{j}_s}{\mathrm{c}} \end{bmatrix}^{-}$$

$$= {}^{-}\begin{bmatrix} E_t\rho + (\boldsymbol{E} - \mathrm{i}c\boldsymbol{B})\bullet\frac{\boldsymbol{j}_s}{\mathrm{c}} & \\ & E_t\frac{\boldsymbol{j}_s}{\mathrm{c}} + (\boldsymbol{E} - \mathrm{i}c\boldsymbol{B})\rho - \mathrm{i}(\boldsymbol{E} - \mathrm{i}c\boldsymbol{B})\times\frac{\boldsymbol{j}_s}{\mathrm{c}} \end{bmatrix}^{+}.$$

$$\vec{e} = {}^{-}\begin{bmatrix} e\gamma & \\ & e\gamma\boldsymbol{\beta} \end{bmatrix}^{-} = \frac{e}{\mathrm{c}}{}^{-}\begin{bmatrix} u_t & \\ & \boldsymbol{u} \end{bmatrix}^{+} = \iiint {}^{-}\begin{bmatrix} \rho & \\ & \frac{\boldsymbol{j}_s}{\mathrm{c}} \end{bmatrix}^{-} \mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

where e is an electric charge, $e\gamma$ is a relativistic charge, $e\gamma\boldsymbol{\beta}$ is a stream charge and $\vec{e}$ is an "en bloc".

Then

$${}^{-}\begin{bmatrix} f_t & \\ & \boldsymbol{f} \end{bmatrix}^{-} = {}^{-}\begin{bmatrix} \frac{\partial}{\partial \mathrm{c}t} & \\ & -\frac{\partial}{\partial \boldsymbol{r}} \end{bmatrix}^{-+}\begin{bmatrix} -\frac{1}{4\pi\varepsilon}\frac{Q}{r} & \\ & 0 \end{bmatrix}^{+}\frac{e}{\mathrm{c}}{}^{-}\begin{bmatrix} u_t & \\ & \boldsymbol{u} \end{bmatrix}^{-}$$

$$= \frac{1}{4\pi\varepsilon}{}^{-}\begin{bmatrix} \frac{\partial}{\partial \mathrm{c}t} & \\ & \frac{\partial}{\partial \boldsymbol{r}}\left(\frac{Q}{r}\right) \end{bmatrix}^{+}\frac{e}{\mathrm{c}}{}^{-}\begin{bmatrix} u_t & \\ & \boldsymbol{u} \end{bmatrix}^{-}$$

$$= \frac{e}{4\pi\varepsilon\mathrm{c}}{}^{-}\begin{bmatrix} \frac{\partial}{\partial \boldsymbol{r}}\left(\frac{Q}{r}\right)\bullet\boldsymbol{u} & \\ & \frac{\partial}{\partial \boldsymbol{r}}\left(\frac{Q}{r}\right)u_t - \mathrm{i}\frac{\partial}{\partial \boldsymbol{r}}\left(\frac{Q}{r}\right)\times\boldsymbol{u} \end{bmatrix}^{-}.$$

The temporal force is

$$f_t = \frac{ke}{\mathrm{c}}\frac{\partial}{\partial \boldsymbol{r}}\left(\frac{Q}{r}\right)\bullet\boldsymbol{u}$$

The spatial force is

$$\boldsymbol{f} = \frac{ke}{\mathrm{c}}\frac{\partial}{\partial \boldsymbol{r}}\left(\frac{Q}{r}\right)u_t - \mathrm{i}\frac{ke}{\mathrm{c}}\frac{\partial}{\partial \boldsymbol{r}}\left(\frac{Q}{r}\right)\times\boldsymbol{u}$$

Here, $\mathrm{k} = \frac{1}{4\pi\varepsilon_0}$ is the Coulomb constant, $k = \frac{1}{4\pi\varepsilon_0\varepsilon_r}$, ε_0 is permittivity of the free-space medium, ε_r is relative permittivity of the dielectric medium.

Moreover, we get the formula between the moment and the impulse that the 4-dimensional force is obtained by integrating time.

$$m\,{}^{-}\begin{bmatrix} \frac{\mathrm{d}ct}{\mathrm{d}\tau} & \\ & \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\tau} \end{bmatrix}^{-} = {}^{-}\begin{bmatrix} mc\gamma & \\ & mc\gamma\boldsymbol{\beta} \end{bmatrix}^{-} = \int_{t_0}^{t}{}^{-}\begin{bmatrix} f_t & \\ & \boldsymbol{f} \end{bmatrix}^{-}\mathrm{d}t.$$

Furthermore, ${}^{-}\begin{bmatrix} f_t & \\ & \boldsymbol{f} \end{bmatrix}^{-} = m\frac{\mathrm{d}}{\mathrm{d}t}{}^{-}\begin{bmatrix} \frac{\mathrm{d}ct}{\mathrm{d}\tau} & \\ & \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\tau} \end{bmatrix}^{-}$,

$$m\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix}\frac{\mathrm{d}ct}{\mathrm{d}\tau} & \\ & \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\tau}\end{bmatrix}=m\frac{\mathrm{d}}{\mathrm{d}ct}\begin{bmatrix}\frac{\mathrm{d}ct}{\mathrm{d}\tau} & \\ & \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\tau}\end{bmatrix}\frac{\mathrm{d}ct}{\mathrm{d}\tau}+\underline{m\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix}\frac{\mathrm{d}ct}{\mathrm{d}\tau} & \\ & \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\tau}\end{bmatrix}\frac{\mathrm{d}r}{\mathrm{d}\tau}+m\frac{\mathrm{d}}{\mathrm{d}\theta}\begin{bmatrix}\frac{\mathrm{d}ct}{\mathrm{d}\tau} & \\ & \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\tau}\end{bmatrix}\frac{\mathrm{d}\theta}{\mathrm{d}\tau}}$$

$$\underline{+m\frac{\mathrm{d}}{\mathrm{d}\varphi}\begin{bmatrix}\frac{\mathrm{d}ct}{\mathrm{d}\tau} & \\ & \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\tau}\end{bmatrix}\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}}.$$

The underlined part is null when the center electric charge is stable.

3. The equation of the electron in atom

$$m_e\frac{\mathrm{d}^2ct}{\mathrm{d}\tau^2}=-\frac{keQ}{c^2r^2}\left(\frac{\boldsymbol{r}}{r}\bullet\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\tau}\right)\frac{\mathrm{d}ct}{\mathrm{d}\tau} \qquad \text{(i)}$$

$$m_e\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}\tau^2}=-\frac{keQ}{c^2r^2}\frac{\boldsymbol{r}}{r}\left(\frac{\mathrm{d}ct}{\mathrm{d}\tau}\right)^2+\mathrm{i}\frac{keQ}{c^2r^2}\left(\frac{\boldsymbol{r}}{r}\times\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\tau}\right)\frac{\mathrm{d}ct}{\mathrm{d}\tau}. \qquad \text{(ii)}$$

The acceleration vector by the spherical coordinate is

$$\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}\tau^2}=\begin{pmatrix}a_r\\a_\theta\\a_\varphi\end{pmatrix}=\begin{pmatrix}\ddot{r}-r\dot{\theta}^2-r\dot{\varphi}^2\sin^2\theta\\2\dot{r}\dot{\theta}+r\ddot{\theta}-r\dot{\varphi}^2\sin\theta\cos\theta\\2\dot{r}\dot{\varphi}\sin\theta+r\ddot{\varphi}\sin\theta+2r\dot{\varphi}\dot{\theta}\cos\theta\end{pmatrix}.$$

We can rewrite the coordinate (t,x,y,z) by the coordinate (t,r,θ,φ).

$$\begin{cases}t=t\\x=r\sin\theta\cos\varphi\\y=r\sin\theta\sin\varphi\\z=r\cos\theta\end{cases}.$$

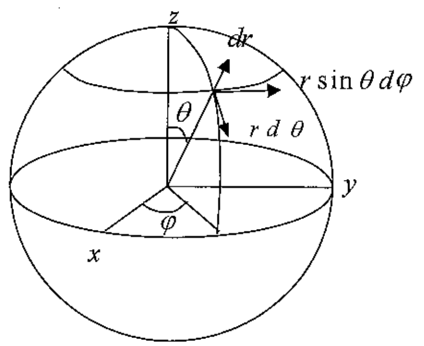

Fig 1. The spherical coordinate.

Theorem 1. The equation of an electron turns around a positive charge.

The relativistic invariant equations of motion by the polar coordinate are

$$\begin{cases}(1)\, m_e\frac{\mathrm{d}^2ct}{\mathrm{d}\tau^2}=-\frac{keQ}{c^2r^2}\frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}ct}{\mathrm{d}\tau}\\(2)\, m_e\frac{\mathrm{d}^2r}{\mathrm{d}\tau^2}=-\frac{keQ}{c^2r^2}\left(\underline{\frac{\mathrm{d}ct}{\mathrm{d}\tau}}\right)^2+\frac{m_e}{r}\left(r\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2+\frac{m_e}{r}\left(r\sin\theta\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)^2\\(3)\, m_e\frac{\mathrm{d}}{\mathrm{d}\tau}\left(r\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)=-\mathrm{i}\frac{keQ}{c^2r^2}\left(\underline{r\sin\theta\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}}\right)\frac{\mathrm{d}ct}{\mathrm{d}\tau}-\frac{m_e}{r}\frac{\mathrm{d}r}{\mathrm{d}\tau}\left(r\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)+m_e\cos\theta\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\left(r\sin\theta\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)\\(4)\, m_e\frac{\mathrm{d}}{\mathrm{d}\tau}\left(r\sin\theta\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)=\mathrm{i}\frac{keQ}{c^2r^2}\left(r\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)\frac{\mathrm{d}ct}{\mathrm{d}\tau}-\frac{m_e}{r}\frac{\mathrm{d}r}{\mathrm{d}\tau}\left(r\sin\theta\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)-m_e\cos\theta\left(r\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}.\end{cases}$$

The above underlined part is related to the Example 2.

The metric is $\mathrm{d}s^2=-\mathrm{d}ct^2+\mathrm{d}r^2+r^2(\sin^2\theta\,\mathrm{d}\varphi^2+d\theta^2)$.

We consider the two-body problem concerned with the nuclear and the electron as in one hydrogen atom. It is assumed that the electron moves on fixed surface. Therefore, we put $\theta=\frac{\pi}{2}-\mathrm{i}\Omega$. Ω is a parameter that relates to the angle of rotation on the orbit.

Then we change the imaginary parts to the real and get a real coefficient equation.

The metric is $\mathrm{d}s^2=-\mathrm{d}ct^2+\mathrm{d}r^2+r^2(\cosh^2\Omega\mathrm{d}\varphi^2-d\Omega^2)$. The coordinate is (t,r,Ω,φ)

$$\begin{cases}ct=ct\\x=r\cosh\Omega\cos\varphi\\y=r\cosh\Omega\sin\varphi\\z=\mathrm{i}r\sinh\Omega\end{cases}.$$

Therefore we get an equation of Newton's type.

Theorem 2. The system of Newton's type.

$$\begin{cases}(1)\, m_e\frac{\mathrm{d}^2ct}{\mathrm{d}\tau^2}=-\frac{keQ}{c^2r^2}\frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}ct}{\mathrm{d}\tau}\\(2)\, m_e\frac{\mathrm{d}^2r}{\mathrm{d}\tau^2}=-\frac{keQ}{c^2r^2}\left(\frac{\mathrm{d}ct}{\mathrm{d}\tau}\right)^2+\frac{m_e}{r}\left\{\left(r\cosh\Omega\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)^2-\left(r\frac{\mathrm{d}\Omega}{\mathrm{d}\tau}\right)^2\right\}\\(3)\, m_e\frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\frac{\mathrm{d}\Omega}{\mathrm{d}\tau}\right)=\left(\frac{keQ}{c^2r^2}\frac{\mathrm{d}ct}{\mathrm{d}\tau}-m_e\sinh\Omega\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)\left(r^2\cosh\Omega\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)\\(4)\, m_e\frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\cosh\Omega\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)=\left(\frac{keQ}{c^2r^2}\frac{\mathrm{d}ct}{\mathrm{d}\tau}-m_e\sinh\Omega\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)\left(r^2\frac{\mathrm{d}\Omega}{\mathrm{d}\tau}\right).\end{cases}$$

We translate the above equations of Newton type into the equations of Kepler type.

The metric is $\mathrm{d}s^2=-c^2\mathrm{d}t^2+\mathrm{d}r^2+r^2(\cosh^2\Omega\mathrm{d}\varphi^2-\mathrm{d}\Omega^2)$.

Theorem 3. The system of equations of Kepler's type.

$$\begin{cases} (1)' \; m_e c\dfrac{\mathrm{d}ct}{\mathrm{d}\tau} = m_e c C_0 e^{\frac{keQ}{m_e c^2 r}} \cdots \text{(the conservation of energy)} \\ (2)' \; \dfrac{\mathrm{d}^2}{\mathrm{d}\tau^2}(r \sinh\Omega) = -\left(\dfrac{keQ}{m_e c^2 r^2}\dfrac{\mathrm{d}ct}{\mathrm{d}\tau}\right)\left(\tanh\Omega - r\cosh\Omega\dfrac{\mathrm{d}\varphi}{\mathrm{d}ct}\right)\cosh\Omega\left(\dfrac{\mathrm{d}ct}{\mathrm{d}\tau}\right) \\ \qquad\qquad \cdots \text{(the structure of space)} \\ (3)' \; r^2\left\{\left(r\cosh\Omega\dfrac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)^2 - \left(r\dfrac{\mathrm{d}\Omega}{\mathrm{d}\tau}\right)^2\right\} = C^2 \cdots \text{(the law of equal areas)} \\ (4)' \; r^2\cosh\Omega\dfrac{\mathrm{d}\varphi}{\mathrm{d}\tau} = C\cosh\Theta'(\geqq 0), \;\; r\dfrac{\mathrm{d}\Omega}{\mathrm{d}\tau} = -C\sinh\Theta' \\ \quad \Theta' = -\displaystyle\int\left(\dfrac{keQ}{m_e c^2 r^2}\dfrac{\mathrm{d}ct}{\mathrm{d}\tau} - \sinh\Omega\dfrac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)\mathrm{d}\tau \cdots\cdots \text{(the internal rotation)} \end{cases}$$

where Ω means being rotational on the orbit.

We call $m_e c C_0 = \dfrac{m_e c^2}{\sqrt{1-\left(\frac{v}{c}\right)^2}} e^{-\frac{keQ}{m_e c^2 r}}$ enegy function.

(Proof)

(1)′ The conservation energy (cf. Example 1 below).

From the equation

$$m_e\frac{\mathrm{d}^2 ct}{\mathrm{d}\tau^2} = -\frac{keQ}{c^2 r^2}\frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}ct}{\mathrm{d}\tau}$$

$$\left(\frac{\mathrm{d}ct}{\mathrm{d}\tau}\right)^{-1}\frac{\mathrm{d}^2 ct}{\mathrm{d}\tau^2} = -\frac{keQ}{m_e c^2 r^2}\frac{\mathrm{d}r}{\mathrm{d}\tau} \therefore \frac{\mathrm{d}}{\mathrm{d}\tau}\log\left(\frac{\mathrm{d}ct}{\mathrm{d}c\tau}\right) = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{keQ}{m_e c^2 r}\right) \therefore \log\left(\frac{\mathrm{d}ct}{\mathrm{d}c\tau}\right) = \frac{keQ}{m_e c^2 r} + c_0$$

$$\therefore \frac{\mathrm{d}ct}{\mathrm{d}c\tau} = e^{\frac{keQ}{m_e c^2 r} + c_0} = e^{c_0} e^{\frac{keQ}{m_e c^2 r}}$$

$$\therefore m_e c^2\frac{\mathrm{d}t}{\mathrm{d}\tau}\left(= \frac{m_e c^2}{\sqrt{1-\left(\frac{v}{c}\right)^2}}\right) = m_e c^2 e^{c_0} e^{\frac{keQ}{m_e c^2 r}} = m_e c. C_0 e^{\frac{keQ}{m_e c^2 r}}, \; c e^{c_0} = C_0$$

Therefore, we get the kinetic energy

$$m_e c\frac{\mathrm{d}ct}{\mathrm{d}\tau} = m_e c C_0 e^{\frac{keQ}{m_e c^2 r}} \qquad (1)'$$

where $e^{\frac{keQ}{m_e c^2 r}}$ is a (extended) potential energy.

(2)′ The structure of space.

By then $(2)\times\sinh\Omega + (3)\times\frac{1}{r}\cosh\Omega$,

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2}\sinh\Omega + \frac{\mathrm{d}}{\mathrm{d}\tau}\left(r\frac{\mathrm{d}\Omega}{\mathrm{d}\tau}\right)\cosh\Omega + \frac{\mathrm{d}r}{\mathrm{d}\tau}\left(\frac{\mathrm{d}\Omega}{\mathrm{d}\tau}\right)\cosh\Omega$$

$$= -\left(\frac{keQ}{m_e c^2 r^2}\frac{\mathrm{d}ct}{\mathrm{d}\tau}\right)\left(\tanh\Omega - r\cosh\Omega\frac{\mathrm{d}\varphi}{\mathrm{d}ct}\right)\cosh\Omega\left(\frac{\mathrm{d}ct}{\mathrm{d}\tau}\right) - \frac{1}{r}\left(r\frac{\mathrm{d}\Omega}{\mathrm{d}\tau}\right)^2\sinh\Omega$$

holds.

Therefore, the structure of space is

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}(r\sinh\Omega) = \frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\frac{\mathrm{d}}{\mathrm{d}\tau}(r\sinh\Omega)\right\} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\frac{\mathrm{d}r}{\mathrm{d}\tau}\sinh\Omega + r\cosh\Omega\frac{\mathrm{d}\Omega}{\mathrm{d}\tau}\right\},$$

$$= \frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2}(\sinh\Omega) + \frac{\mathrm{d}r}{\mathrm{d}\tau}\cosh\Omega\frac{\mathrm{d}\Omega}{\mathrm{d}\tau} + \sinh\Omega\frac{\mathrm{d}\Omega}{\mathrm{d}\tau}\left(r\frac{\mathrm{d}\Omega}{\mathrm{d}\tau}\right) + \frac{\mathrm{d}}{\mathrm{d}\tau}\left(r\frac{\mathrm{d}\Omega}{\mathrm{d}\tau}\right)\cosh\Omega,$$

$$= -\left(\frac{keQ}{m_e c^2 r^2}\frac{\mathrm{d}ct}{\mathrm{d}\tau}\right)\left(\tanh\Omega - r\cosh\Omega\frac{\mathrm{d}\varphi}{\mathrm{d}ct}\right)\cosh\Omega\left(\frac{\mathrm{d}ct}{\mathrm{d}\tau}\right). \qquad (2)'$$

(3)′ The law of equal areas.

By the $(4)\times\left(r^2\cosh\Omega\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right) - (3)\times\left(r^2\frac{\mathrm{d}\Omega}{\mathrm{d}\tau}\right)$

We get $r^2\cosh\Omega\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\cdot\frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\cosh\Omega\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right) - r^2\frac{\mathrm{d}\Omega}{\mathrm{d}\tau}\cdot\frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\frac{\mathrm{d}\Omega}{\mathrm{d}\tau}\right) = 0$

Therefore

$$r^2\underline{\underline{\left\{\left(r\cosh\Omega\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)^2 - \left(r\frac{\mathrm{d}\Omega}{\mathrm{d}\tau}\right)^2\right\}}} = C^2.$$

(4)′ The rotation on the orbit.

From (3)′, we can put $r^2\cosh\Omega\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = C\cosh\Theta'(\geqq 0)$, $r^2\frac{\mathrm{d}\Omega}{\mathrm{d}\tau} = -C\sinh\Theta'$.

Then $\dfrac{r^2\frac{\mathrm{d}\Omega}{\mathrm{d}\tau}}{r^2\cosh\Omega\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}} = -\tanh\Theta'$ holds. (Cf. $\frac{\mathrm{d}r}{\mathrm{d}ct} = \tanh\Theta$.)

Therefore, from the equation (3) + (4), we get

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(\sinh\Theta' + \cosh\Theta') = -\left(\frac{keQ}{m_e c^2 r^2}\frac{\mathrm{d}ct}{\mathrm{d}\tau} - \sinh\Omega\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)(\cosh\Theta' + \sinh\Theta').$$

$$\frac{\mathrm{d}e^{\Theta'}}{\mathrm{d}\tau} = -\left(\frac{keQ}{m_e c^2 r^2}\frac{\mathrm{d}ct}{\mathrm{d}\tau} - \sinh\Omega\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)e^{\Theta'}$$

$$\therefore \frac{\mathrm{d}\Theta'}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau}\log e^{\Theta'} = -\left(\frac{keQ}{m_e c^2 r^2}\frac{\mathrm{d}ct}{\mathrm{d}\tau} - \sinh\Omega\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right). \qquad (3)'$$

The internal rotation is Eq. (4).

$$\therefore \Theta' = -\int\left(\frac{keQ}{m_e c^2 r^2}\frac{\mathrm{d}ct}{\mathrm{d}\tau} - \sinh\Omega\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)\mathrm{d}\tau. \qquad (4)'$$

(Q.E.D.)

4. The Examples

(Example 1.) The difference of energy between two orbits.

This equation (1)′ in this system means the law of the conservation of energy because the energy function is

$$m_e c C_0 = \frac{m_e c^2}{\sqrt{1-\left(\frac{v}{c}\right)^2}} e^{-\frac{keQ}{m_e c^2 r}}$$

$$= m_e c^2\left(1+\frac{1}{2}\frac{v^2}{c^2}+\cdots\right)\left(1-\frac{keQ}{m_e c^2 r}+\frac{1}{2}\left(\frac{keQ}{m_e c^2 r}\right)^2+\cdots\right)$$

$$= m_e c^2 + \underline{\underline{\frac{1}{2} m_e v^2}} - \underline{\underline{\frac{keQ}{r}}} + \cdots$$

where the underlined part is the kinetic energy and the potential energy.

We consider the case of a hydrogen atom and the surrounding electron orbit is a circle.

By the formula Theorem 2. (2) When $\frac{d^2 r}{d\tau^2} \equiv 0$ (the circle orbit is), then

$$m_e \frac{d^2 r}{d\tau^2} = -\frac{keQ}{c^2 r^2}\left(\frac{dct}{d\tau}\right)^2 + \frac{m_e}{r}\left\{\left(r \cosh\Omega \frac{d\varphi}{d\tau}\right)^2 - \left(r\frac{d\Omega}{d\tau}\right)^2\right\}$$

Therefore we get the equation of the balance:

$$\underline{\underline{\frac{keQ}{c^2 r^2}}} \text{(The Coulomb's law)} = \frac{m_e}{r}\left\{\left(r \cosh\Omega \frac{d\varphi}{dct}\right)^2 - \left(r\frac{d\Omega}{dct}\right)^2\right\} \text{(The acceleration)}$$

$$= \frac{m_e}{r}\left(r\frac{d\Phi}{dct}\right)^2 = \underline{\underline{\frac{m_e v^2}{c^2 r}}} \qquad \text{(A)}.$$

Then the energy function on the circle orbit is $m_e c C_0 = \frac{m_e c^2}{\sqrt{1-\left(\frac{v}{c}\right)^2}} e^{-\frac{keQ}{m_e c^2 r}} = m_e c^2 - \frac{1}{2}\frac{keQ}{r} + \cdots$

(Example 2.) the minimum radius.

We take the relativistic moment $P = \frac{m_e c \frac{v}{c}}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$, then the quantum condition is a

$$2\pi r_n = n\lambda_e = \frac{n h\sqrt{1-\left(\frac{v_n}{c}\right)^2}}{m_e v_n}, \text{ where } r_n, v_n \text{ is depend on } n. \qquad \text{(B)}.$$

Then $(2\pi r_n c m_e v_n)^2 = (nh)^2(c^2 - v_n^2)$.

Moreover, this formula is substituted for the (A) $v_n^2 = \frac{keQ}{m_e r_n}$.

Then $(2\pi r_n c m_e)^2 \frac{keQ}{m_e r_n} = (nh)^2\left(c^2 - \frac{keQ}{m_e r_n}\right)$

$keQ(2\pi c m_e)^2 r_n^2 - (n ch)^2 m_e r_n + keQ(nh)^2 = 0$.

Therefore we get the orbital radius of electron

$$r_n = \frac{(n ch)^2 m_e \pm \sqrt{\{(n ch)^2 m_e\}^2 - 4keQ(2\pi c m_e)^2 keQ(nh)^2}}{2k_e Q(2\pi c m_e)^2}$$

$$= \frac{(nh)^2\left(1+\sqrt{1-\left(\frac{4\pi keQ}{nhc}\right)^2}\right)}{2(2\pi)^2 keQ m_e} \left(\text{Cf. } r_n = \frac{(nh)^2}{(2\pi)^2 keQ m_e} \text{ in classical}\right).$$

Especially $n=1$, it is obtained that $r_1 = \frac{h^2\left(1+\sqrt{1-\left(\frac{4\pi keQ}{hc}\right)^2}\right)}{2(2\pi)^2 keQ m_e} = 5.29137856 \times 10^{-11}$

$$\left(\text{Cf. } r_1 = \frac{h^2}{(2\pi)^2 keQ m_e} = 5.29166 \times 10^{-11} \text{ in classical}\right).$$

Moreover the formula of r_n is indicated the limited radius in a circle orbit.

That is to say, when the case $\left(\frac{4\pi keQ}{nhc}\right)^2 = 1$, then $n = \frac{4\pi keQ}{hc} = \frac{1}{68.5165}$, therefore

$$r_t = \frac{(nh)^2}{2(2\pi)^2 keQ m_e} = \frac{2keQ}{m_e c^2} = 5.636 \times 10^{-15} \text{ (the limited radius)}.$$

And this value is obtained also by the Energy function.

$$F(r)(= m_e c C_0) = \frac{m_e c^2}{\sqrt{1-\frac{keQ}{m_e c^2 r}}} e^{-\frac{keQ}{m_e c^2 r}}, \text{ because}$$

$$F'(r) = \frac{1}{2}\frac{m_e c^2}{\sqrt{\left(1-\frac{keQ}{m_e c^2 r}\right)^3}} \frac{keQ}{m_e c^2 r^2} e^{-\frac{keQ}{m_e c^2 r}}\left(1-\frac{2keQ}{m_e c^2 r}\right) = 0.$$

Therefore, the minimum point is $r = \frac{2keQ}{m_e c^2}$.

(Example 3.) acceleration.

The electronic movement is mostly determined by the proton electric charge of a central nucleus.

Then its acceleration is

$$\begin{bmatrix} f_t \\ f \end{bmatrix}$$

$$= \frac{\mathrm{d}}{\mathrm{d}ct}\begin{bmatrix} E \\ \mathrm{c}(p_x, p_y, p_z) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ (E_x, 0, 0) \end{bmatrix}^{+} e \begin{bmatrix} \frac{u_0}{\mathrm{c}} \\ \left(0, \frac{u_y}{\mathrm{c}}, 0\right) \end{bmatrix}_{(x,0,0)}$$

$$\frac{u_0}{\mathrm{c}} = \gamma = \frac{\mathrm{d}ct}{\mathrm{d}c\tau}, \quad \frac{u_y}{\mathrm{c}} = \gamma\beta_y = \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}y}{\mathrm{d}ct} = \frac{\mathrm{d}y}{\mathrm{d}c\tau}$$

$$= e\begin{bmatrix} 0 \\ (E_x\gamma, 0, -\mathrm{i}E_x\gamma\beta) \end{bmatrix}, -eE_x = \frac{keQ}{r^2}\left(= \frac{m_e v_y^2}{r} = m_e v_y \cdot \frac{v_y}{r}\right).$$

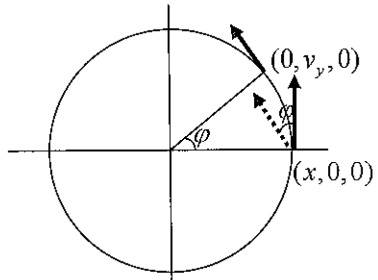

Fig 2. Schematic diagram of the electronic movement.

Therefore the 4-dimensional force (f_t, f_x, f_y, f_z) on the electron is $f_t = 0$.

$$f_x = eE_x\gamma = -\frac{keQ}{r^2}\underline{\underline{\frac{1}{\sqrt{1-\left(\frac{v_y}{\mathrm{c}}\right)^2}}}} = -\frac{m_e v_y^2}{r}\frac{1}{\sqrt{1-\left(\frac{v_y}{\mathrm{c}}\right)^2}} = -\frac{m_e v_y}{\sqrt{1-\left(\frac{v_y}{\mathrm{c}}\right)^2}}\frac{v_y}{r} = -P_x\frac{\mathrm{d}\varphi}{\mathrm{d}t}, \quad P_x = \frac{m_e v_y}{\sqrt{1-\left(\frac{v_y}{\mathrm{c}}\right)^2}}.$$

$$f_y = 0, \quad f_z = -\mathrm{i}eE_x\gamma\beta = -\mathrm{i}\frac{keQ}{r^2}\underline{\underline{\frac{\frac{v_y}{\mathrm{c}}}{\sqrt{1-\left(\frac{v_y}{\mathrm{c}}\right)^2}}}} = -\mathrm{i}\frac{m_e\frac{v_y^2}{\mathrm{c}}}{\sqrt{1-\left(\frac{v_y}{\mathrm{c}}\right)^2}}\frac{v_y}{r} = -\mathrm{i}P_z\frac{\mathrm{d}\varphi}{\mathrm{d}t}, \quad P_z = \frac{m_e v_y}{\sqrt{1-\left(\frac{v_y}{\mathrm{c}}\right)^2}}\frac{v_y}{\mathrm{c}},$$

where $\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{v_y}{r} = \sqrt{\frac{keQ}{m_e r^3}}$ is an angular velocity. (Cf. **Theorem 1.**)

When radius r is a Bohr radius r_1, we look upon $P_x(=p_y) = \frac{m_e \mathrm{c}\frac{v_y}{\mathrm{c}}}{\sqrt{1-\left(\frac{v_y}{\mathrm{c}}\right)^2}}$ as $\frac{\mathrm{h}}{\lambda_e}$, where h is the Planck constant and λ_e is the circumferential length.

$$\frac{\mathrm{h}}{\lambda_e} = \frac{m_e v_y}{\sqrt{1-\left(\frac{v_y}{\mathrm{c}}\right)^2}}, \quad \mathrm{h} = \lambda_e\frac{m_e v_y}{\sqrt{1-\left(\frac{v_y}{\mathrm{c}}\right)^2}} = \frac{2\pi r m_e v_y}{\sqrt{1-\left(\frac{v_y}{\mathrm{c}}\right)^2}}.$$

We give this value h an another meaning as the angular moment of electron on the Bohr orbit.

Moreover, $P_z\left(= p_y\frac{v_y}{\mathrm{c}}\right) = \frac{m_e \mathrm{c}\frac{v_y^2}{\mathrm{c}^2}}{\sqrt{1-\left(\frac{v_y}{\mathrm{c}}\right)^2}}$

$\frac{P_z}{P_x} = \frac{v_y}{\mathrm{c}} = \frac{\lambda_e\nu_e}{\mathrm{c}}, \ P_z = \frac{\lambda_e\nu_e}{\mathrm{c}}P_x = \frac{\lambda_e\nu_e}{\mathrm{c}}\frac{\mathrm{h}}{\lambda_e} = \frac{\mathrm{h}\nu_e}{\mathrm{c}}$. Therefore we put $E' = \mathrm{c}P_z = \mathrm{h}\nu_e$ and we call this imaginary moment a surrounding frequency energy, where $\lambda_e = 2\pi r$ and $2\pi r\nu_e = v_y$

Conclusion

The authors calculate the relativistic energy in the orbit of the electron in atom. These relation images suggest to us a figure of the electron in atom. We get that the similarity between the Planck constant and the angular moment and see the relation between the surrounding frequency energy and the angular moment.

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