~~重力方程式と電磁力~~

~~(相対論的不変性を保持して)~~

The Equation of Gravitational Force

and

the Electromagnetic Force

(under the relativistic invariant)

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In this paper, we discuss the deduction of 4-dimensional equation of motion which is relativistic invariant.

Contents:

In §1 for preliminaries we mention the modified Maxwell's equation in which we have the time-component of electromagnetic field and use the **matrix-vector** and **relativistic form**[1).]

In §2 we consider two forces which is caused by a charge and a mass respectively. These forces are similar in the inverse square law. We improve and push forward the similarity to the potential, field and force.

In §3 we can deduce the 4-dimensional equation of motion which is relativistic

invariant. And in the following paper, this equation contains Kepler's Law and its complex components explain the relativistic effect.

§1. Coulomb-Lorentz Force

In the previous paper[1), we can represented Maxwell's equation and its force as a 4-dimensional matrix vector.

Let $\mathbb{E}=\mathbf{E}-ic\mathbf{B}$ be an electric and magnetic field as a complex 3-dimensional field in space and $E_t - icB_t$ ($B_t = 0$) the time-component.

Then we have a relation of **a matrix-vector** between 4-dimensional potential $(\phi, \mathbf{A})$ and electromagnetic field $(E_t, \mathbf{E}-ic\mathbf{B})$ as follows:

$$
{}^{-}\begin{pmatrix} E_t & \\ & \mathbf{E}-ic\mathbf{B} \end{pmatrix}^{+} = {}^{-}\begin{pmatrix} \partial ct & \\ & -\partial\mathbf{r} \end{pmatrix}^{-}\ {}^{+}\begin{pmatrix} \phi & \\ & -c\mathbf{A} \end{pmatrix}^{+}
$$

$$
= {}^{-}\begin{pmatrix} \dfrac{\partial\phi}{\partial ct}+divc\mathbf{A} & \\ & -\dfrac{\partial c\mathbf{A}}{\partial ct}-\mathbf{grad}\phi - i\mathbf{rot}c\mathbf{A} \end{pmatrix}^{+} \quad \cdots (*).
$$

Where signs "+", "−" mean relativistic invariant [1).

We compare the components of this relation, then.

$$
\begin{cases}
\underline{E_t} = \dfrac{\partial\phi}{\partial ct} + divc\mathbf{A} \cdots (1)' \\
\mathbf{E} = -\dfrac{\partial c\mathbf{A}}{\partial ct} - \mathbf{grad}\phi \cdots (2) \\
c\mathbf{B} = \mathbf{rot}c\mathbf{A} \cdots (3)
\end{cases}.
$$

Where the above underlined part is a **time-component**.

And we have a Lorenz gauge $E_t = \dfrac{\partial\phi}{\partial ct} + divc\mathbf{A} = 0$ and a Coulomb gauge $E_t = \dfrac{\partial\phi}{\partial ct} (\Leftrightarrow divc\mathbf{A} = 0)$.

And the Maxwell's equation is as follows:

$$
{}^{+}\begin{pmatrix} \rho & \\ & -\mathbf{j} \end{pmatrix}^{+} = {}^{+}\begin{pmatrix} \partial ct & \\ & \partial\mathbf{r} \end{pmatrix}^{+}\ {}^{-}\begin{pmatrix} E_t & \\ & \mathbf{E}-ic\mathbf{B} \end{pmatrix}^{+}
$$

$$= {}^{+}\begin{pmatrix} \frac{\partial E_t}{\partial ct} + div(\mathbf{E} - ic\mathbf{B}) & \\ & \frac{\partial(\mathbf{E} - ic\mathbf{B})}{\partial ct} + \mathbf{grad}E_t - i\mathbf{rot}(\mathbf{E} - ic\mathbf{B}) \end{pmatrix}^{+},$$

$$\begin{cases} \mathbf{rotE} + \frac{\partial c\mathbf{B}}{\partial ct} = \mathbf{0} \cdots (4) \\ divc\mathbf{B} = 0 \cdots (5) \\ div\mathbf{E} + \underline{\frac{\partial E_t}{\partial ct}} = \rho \cdots (6)' \\ \mathbf{rot}c\mathbf{B} - \frac{\partial \mathbf{E}}{\partial ct} - \underline{\mathbf{grad}E_t} = \mathbf{j} \cdots (7)' \end{cases}.$$

Where the above underlined part is a derivative of **time-component**.

Therefore the Coulomb-Lorentz force to the moving charge in electromagnetic field is as follows:

$${}^{-}\begin{pmatrix} F_t & \\ & \mathbf{F} \end{pmatrix}^{-} = {}^{-}\begin{pmatrix} E_t & \\ & \mathbf{E} - ic\mathbf{B} \end{pmatrix}^{+} {}^{-}\begin{pmatrix} q & \\ & \mathbf{j} \end{pmatrix}^{-}$$

$$= {}^{-}\begin{pmatrix} E_t q + (\mathbf{E} - ic\underline{\mathbf{B}}) \cdot \mathbf{j} & \\ & E_t \mathbf{j} + (\mathbf{E} - i\underline{c\mathbf{B}}) \cdot q - i(\underline{\mathbf{E}} - ic\mathbf{B}) \times \mathbf{j} \end{pmatrix}^{-} \cdots (**),$$

$$\begin{cases} F_t = qE_t + \mathbf{j} \cdot \mathbf{E} - i\underline{\mathbf{j} \cdot c\mathbf{B}} & \text{(the variation of energy)} \\ \mathbf{F} = q\mathbf{E} + \mathbf{j}E_t + \mathbf{j} \times c\mathbf{B} - i(\underline{qc\mathbf{B} - \mathbf{j} \times \mathbf{E}}) & \text{(the variation of momentum)} \end{cases}.$$

Where the above underlined part is a **complex force**.

§2. Coulomb-Lorentz force and gravitational one

We consider the 4-dimensional potential $\phi(x, y, z) = -\frac{1}{4\pi\varepsilon_0}\frac{e}{r}$ (ε_0 is a dielectric constant) and $A(x, y, z) = 0$ which are caused by the stationary (**negative**) charge "−e".

Then the 4-dimensional electromagnetic field $(E_t, \mathbf{E} - ic\mathbf{B})$ is given by the above formula (*),

$$
{}^{-}\begin{pmatrix} E_t & \\ & \mathbf{E}-ic\mathbf{B} \end{pmatrix}^{+} = {}^{-}\begin{pmatrix} \partial ct & \\ & -\partial\mathbf{r} \end{pmatrix}^{-} {}^{+}\begin{pmatrix} -\frac{1}{4\pi\varepsilon_0}\frac{e}{r} & \\ & 0 \end{pmatrix}^{+} = \frac{1}{4\pi\varepsilon_0}\,{}^{-}\begin{pmatrix} 0 & \\ & \frac{\partial}{\partial\mathbf{r}}(\frac{e}{r}) \end{pmatrix}^{+}.
$$

That is, the electric field is

$$
\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{\partial}{\partial\mathbf{r}}(\frac{e}{r}) = -\frac{e}{4\pi\varepsilon_0 r^2}\frac{\mathbf{r}}{r}.
$$

And the magnetic field and the time-component are

$\mathbf{B} = \mathbf{0}$ and $E_t = 0$.

And we put $(q,\mathbf{j}) = (q_0\gamma, q_0\gamma\boldsymbol{\beta}) = (\frac{q_0}{c}u_t, \frac{q_0}{c}\mathbf{u})$ where $u_t = \frac{dct}{d\tau} = c\gamma$, $\mathbf{u} = \frac{d\mathbf{r}}{d\tau} = c\gamma\boldsymbol{\beta}$.

Then by the above formula (**), the **Coulomb-Lorentz force** which acts on the moving charge $(q,\mathbf{j})$ in the electromagnetic field is

$$
{}^{-}\begin{pmatrix} F_t & \\ & \mathbf{F} \end{pmatrix}^{-} = {}^{-}\begin{pmatrix} \partial ct & \\ & -\partial\mathbf{r} \end{pmatrix}^{-} {}^{+}\begin{pmatrix} -\frac{1}{4\pi\varepsilon_0}\frac{e}{r} & \\ & 0 \end{pmatrix}^{+} \frac{q_0}{c}\,{}^{-}\begin{pmatrix} u_t & \\ & \mathbf{u} \end{pmatrix}^{-}
$$

$$
= \frac{1}{4\pi\varepsilon_0}\,{}^{-}\begin{pmatrix} 0 & \\ & \frac{\partial}{\partial\mathbf{r}}(\frac{e}{r}) \end{pmatrix}^{+} \frac{q_0}{c}\,{}^{-}\begin{pmatrix} u_t & \\ & \mathbf{u} \end{pmatrix}^{-}
$$

$$
= \frac{q_0}{4\pi\varepsilon_0 c}\,{}^{-}\begin{pmatrix} \frac{\partial}{\partial\mathbf{r}}(\frac{e}{r})\cdot\mathbf{u} & \\ & \frac{\partial}{\partial\mathbf{r}}(\frac{e}{r})u_t - i\frac{\partial}{\partial\mathbf{r}}(\frac{e}{r})\times\mathbf{u} \end{pmatrix}^{-} \quad \cdots (***),
$$

$$
\begin{cases} F_t = \frac{q_0}{4\pi\varepsilon_0 c}\frac{\partial}{\partial\mathbf{r}}(\frac{e}{r})\cdot\mathbf{u} & \text{(the variation of energy)} \\ \mathbf{F} = \frac{q_0}{4\pi\varepsilon_0 c}\frac{\partial}{\partial\mathbf{r}}(\frac{e}{r})u_t - i\underline{\frac{q_0}{4\pi\varepsilon_0 c}\frac{\partial}{\partial\mathbf{r}}(\frac{e}{r})\times\mathbf{u}} & \text{(the variation of momentum)} \end{cases}.
$$

The above underlined part is a **complex force**.

We compare this force and the gravitational one which is caused by the stationary mass "M" (for simplicity) as follows:

The relation of its potential $U = \frac{G}{c^2}\frac{M}{r}$ and gravitation force $\mathbf{f}$ is

$$\mathbf{f} = m_0 \frac{\partial U}{\partial \mathbf{r}} \fallingdotseq m_0 \gamma \frac{\partial U}{\partial \mathbf{r}} = \frac{G m_0}{c^2} \frac{\partial}{\partial \mathbf{r}} (\frac{M}{r}) u_t .$$

Where $\gamma = \frac{u_t}{c} = \frac{dct}{cd\tau} = \frac{1}{\sqrt{1-(\frac{v}{c})^2}} (\fallingdotseq 1)$, G is a gravitational constant and c is a light velocity.

This gravitational force $\mathbf{f}$ is quite similar to the real part of the Coulomb-Lorentz one $\mathbf{F} = \frac{q_0}{4\pi\varepsilon_0 c} \frac{\partial}{\partial \mathbf{r}} (\frac{e}{r}) u_t$ −i(imaginary part).

Therefore, we get the 4-dimentional force $(f_t, \mathbf{f})$ which is caused by the stationary mass "M", that is, the potential is

$${}^{+}\begin{pmatrix} U & \\ & \mathbf{0} \end{pmatrix}^{+} = {}^{+}\begin{pmatrix} -\frac{G}{c^2}\frac{M}{r} & \\ & 0 \end{pmatrix}^{+}$$

which is corresponding to ${}^{+}\begin{pmatrix} -\frac{1}{4\pi\varepsilon_0}\frac{e}{r} & \\ & 0 \end{pmatrix}^{+}$.

And its gravitational field is

$${}^{-}\begin{pmatrix} \partial ct & \\ & -\partial \mathbf{r} \end{pmatrix}^{-} {}^{+}\begin{pmatrix} -\frac{G}{c^2}\frac{M}{r} & \\ & 0 \end{pmatrix}^{+} = \frac{G}{c^2} {}^{-}\begin{pmatrix} 0 & \\ & \frac{\partial}{\partial \mathbf{r}}(\frac{M}{r}) \end{pmatrix}^{+} .$$

Therefore we get the 4-dimentional gravitational force as follows;

$${}^{-}\begin{pmatrix} f_t & \\ & \mathbf{f} \end{pmatrix}^{-} = {}^{-}\begin{pmatrix} \partial ct & \\ & -\partial \mathbf{r} \end{pmatrix}^{-} {}^{+}\begin{pmatrix} -\frac{G}{c^2}\frac{M}{r} & \\ & 0 \end{pmatrix}^{+} \frac{q_0}{c} {}^{-}\begin{pmatrix} u_t & \\ & \mathbf{u} \end{pmatrix}^{-}$$

$$= \frac{G}{c^2} {}^{-}\begin{pmatrix} 0 & \\ & \frac{\partial}{\partial \mathbf{r}}(\frac{M}{r}) \end{pmatrix}^{+} \frac{m_0}{c} {}^{-}\begin{pmatrix} u_t & \\ & \mathbf{u} \end{pmatrix}^{-}$$

$$= \frac{G m_0}{c^3} {}^{-}\begin{pmatrix} \frac{\partial}{\partial \mathbf{r}}(\frac{M}{r}) \cdot \mathbf{u} & \\ & \frac{\partial}{\partial \mathbf{r}}(\frac{M}{r}) u_t - i \frac{\partial}{\partial \mathbf{r}}(\frac{M}{r}) \times \mathbf{u} \end{pmatrix}^{-} .$$

That is,

$$\begin{cases} f_t = \dfrac{Gm_0}{c^3}\dfrac{\partial}{\partial \mathbf{r}}(\dfrac{M}{r})\cdot \mathbf{u} & \text{(the variation of energy)} \\ \mathbf{f} = \dfrac{Gm_0}{c^3}\dfrac{\partial}{\partial \mathbf{r}}(\dfrac{M}{r})u_t - i\underline{\dfrac{Gm_0}{c^3}\dfrac{\partial}{\partial \mathbf{r}}(\dfrac{M}{r})\times \mathbf{u}} & \text{(the variation of momentum)} \end{cases} .$$

The above underlined part is a **complex force** and its interpretation is in the following paper.

§3. The 4-dimensional equation of motion which is relativistic invariant

In the above discussion, we had correspond the source (**negative**) charge "$-\mathbf{e}$" to the source mass M, the moving charge $(q,\mathbf{j}) = (q_0\gamma, q_0\gamma\beta)$ to the moving mass $(\mathbf{m}_0\gamma, \mathbf{m}_0\gamma\beta)$ and the constant $\dfrac{\mathbf{1}}{\mathbf{4\pi\varepsilon_0}}$ of the Coulomb-Lorentz force to the gravitational constant $\dfrac{\mathbf{G}}{c^2}$.

Then we get the modified equation of motion.

<u>Theorem 1</u>

The equation of motion which is relativistic invariant is

$$\begin{cases} \dfrac{d^2ct}{d\tau^2} = -\dfrac{M_G}{r^2}(\dfrac{\mathbf{r}}{r}\cdot\dfrac{d\mathbf{r}}{d\tau})\dfrac{dct}{d\tau}\cdots(1)_{ct} \\ \dfrac{d^2\mathbf{r}}{d\tau^2} = -\dfrac{M_G}{r^2}\dfrac{\mathbf{r}}{r}(\dfrac{dct}{d\tau})^2 + i\dfrac{M_G}{r^2}(\dfrac{\mathbf{r}}{r}\times\dfrac{d\mathbf{r}}{d\tau})\dfrac{dct}{d\tau}\cdots(2)_r + i\{(3)_\theta + (4)_\phi\} \end{cases} .$$

Proof

We replace $\dfrac{Q}{r} = -\dfrac{e}{4\pi\varepsilon_0 r}$ (potential of "**negative**" **stationary** charge), $(q,\mathbf{j})$ which is "**positive moving** charge" as $-\dfrac{M_G}{r} = -\dfrac{GM}{c^2 r}$ (potential of **stationary** mass), $(\mathbf{m}_0\gamma, \mathbf{m}_0\gamma\beta)$ which is "**moving** mass" in the formula (***).

And by this replacement, we get the 4-dimensional gravitational force as follows:

$$^{-}\begin{pmatrix} f_t & \\ & \mathbf{f} \end{pmatrix}^{-} = \underline{^{-}\begin{pmatrix} \partial ct & \\ & -\partial\mathbf{r} \end{pmatrix}^{-} {}^{+}\begin{pmatrix} -\dfrac{M_G}{r} & \\ & \mathbf{0} \end{pmatrix}^{+}} \dfrac{m_0}{c}\, {}^{-}\begin{pmatrix} u_t & \\ & \mathbf{u} \end{pmatrix}^{-} 、 \dfrac{M_G}{r} = \dfrac{GM}{c^2 r} .$$

Where the underlined part is a 4-dimensional gravitational field.

And we integrate this formula by time then

$$\int_{t_0}^{t} {}^{-}\begin{pmatrix} f_t & \\ & \mathbf{f} \end{pmatrix}^{-} cdt = \int_{t_0}^{t} {}^{-}\begin{pmatrix} \partial ct & \\ & -\partial \mathbf{r} \end{pmatrix}^{-} {}^{+}\begin{pmatrix} -\dfrac{M_G}{r} & \\ & \mathbf{0} \end{pmatrix}^{+} \frac{m_0}{c} {}^{-}\begin{pmatrix} u_t & \\ & \mathbf{u} \end{pmatrix}^{-} cdt$$

means a variation of energy-momentum

$$\left[{}^{-}\begin{pmatrix} m_0\gamma & \\ & m_0\gamma\boldsymbol{\beta} \end{pmatrix}^{-} \right]_{t_0}^{t} .$$

Therefore we get the modified equation of motion as follows;

$$m_0 \frac{d}{d\tau} {}^{-}\begin{pmatrix} \dfrac{dct}{d\tau} & \\ & \dfrac{d\mathbf{r}}{d\tau} \end{pmatrix}^{-} = c\frac{d}{d\tau} {}^{-}\begin{pmatrix} m_0\gamma & \\ & m_0\gamma\boldsymbol{\beta} \end{pmatrix}^{-}$$

$$= -M_G m_0 {}^{-}\begin{pmatrix} \partial ct & \\ & -\partial \mathbf{r} \end{pmatrix}^{-} {}^{+}\begin{pmatrix} \dfrac{1}{r} & \\ & \mathbf{0} \end{pmatrix}^{+} {}^{-}\begin{pmatrix} \dfrac{dct}{d\tau} & \\ & \dfrac{d\mathbf{r}}{d\tau} \end{pmatrix}^{-} \frac{dct}{d\tau}$$

$$= -\frac{M_G m_0}{r^2} {}^{-}\begin{pmatrix} (\dfrac{\mathbf{r}}{r}\cdot\dfrac{d\mathbf{r}}{d\tau})(\dfrac{dct}{d\tau}) & \\ & \dfrac{\mathbf{r}}{r}(\dfrac{dct}{d\tau})^2 - i(\dfrac{\mathbf{r}}{r}\times\dfrac{d\mathbf{r}}{d\tau})(\dfrac{dct}{d\tau}) \end{pmatrix}^{-} .$$

Q.E.D.

We can rewrite the coordinate (x, y, z) by the spherical polar coordinate (r, θ, ϕ) ,that is,

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}.$$

Then we get

<u>Corollary 2</u>

The equation of motion at the spherical polar coordinate is

$$\begin{cases} \dfrac{d^2ct}{d\tau^2} = -\dfrac{M_G}{r^2}\dfrac{dr}{d\tau}\dfrac{dct}{d\tau} \cdots (1)_{ct} \\ \dfrac{d^2r}{d\tau^2} = -\dfrac{M_G}{r^2}(\dfrac{dct}{d\tau})^2 + \dfrac{1}{r}(r\dfrac{d\theta}{d\tau})^2 + \dfrac{1}{r}(r\sin\theta\dfrac{d\phi}{d\tau})^2 \cdots (2)_r \\ \dfrac{d}{d\tau}(r\dfrac{d\theta}{d\tau}) = -i\dfrac{M_G}{r^2}(r\sin\theta\dfrac{d\phi}{d\tau})\dfrac{dct}{d\tau} - \dfrac{1}{r}\dfrac{dr}{d\tau}(r\dfrac{d\theta}{d\tau}) + \cos\theta\dfrac{d\phi}{d\tau}(r\sin\theta\dfrac{d\phi}{d\tau}) \cdots i(3)_\theta \\ \dfrac{d}{d\tau}(r\sin\theta\dfrac{d\phi}{d\tau}) = i\dfrac{M_G}{r^2}(r\dfrac{d\theta}{d\tau})\dfrac{dct}{d\tau} - \dfrac{1}{r}\dfrac{dr}{d\tau}(r\sin\theta\dfrac{d\phi}{d\tau}) - \cos\theta(r\dfrac{d\theta}{d\tau})\dfrac{d\phi}{d\tau} \cdots i(4)_\phi \end{cases}.$$

Proof:

$(1)_{ct}$: Formula (1) is the same one

And by the proposition 3 below, we get the formulas $(2)_r, (3)_\theta, (4)_\phi$ as follows:

By the theorem 1

$$\frac{d^2\mathbf{r}}{d\tau^2} = -\frac{M_G}{r^2}\frac{\mathbf{r}}{r}(\frac{dct}{d\tau})^2 + i\frac{M_G}{r^2}(\frac{\mathbf{r}}{r}\times\frac{d\mathbf{r}}{d\tau})\frac{dct}{d\tau}\cdot\ \cdot\ \cdot (2)_r + i\{(3)_\theta + (4)_\phi\},$$

$$\frac{d\mathbf{r}}{d\tau} = \begin{pmatrix} v_r \\ v_\theta \\ v_\phi \end{pmatrix} = \begin{pmatrix} \dot{r} \\ r\dot{\theta} \\ r\sin\theta\dot{\phi} \end{pmatrix} \text{ and } \frac{\mathbf{r}}{r}\times\frac{d\mathbf{r}}{d\tau} = \begin{pmatrix} 0 \\ -r\sin\theta\dot{\phi} \\ r\dot{\theta} \end{pmatrix}.$$

$(2)_r$: The component of $\mathbf{r}$- direction is

$$\alpha_r = \frac{d^2\mathbf{r}}{d\tau^2}\bullet\frac{\mathbf{r}}{|\mathbf{r}|} = -\frac{M_G}{r^2}(\frac{dct}{d\tau})^2 \text{ (where “}\bullet\text{” is an inner product.).}$$

Therefore by the proposition 3. (α_r) ,we get

$$-\frac{M_G}{r^2}(\frac{dct}{d\tau})^2 = \ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta$$

$$= \frac{d^2 r}{d\tau^2} - \frac{1}{r}(r\frac{d\theta}{d\tau})^2 - \frac{1}{r}(r\sin\theta\frac{d\phi}{d\tau})^2 .$$

$(3)_\theta$: The component of $\mathbf{rd\theta}$ direction is

$$\alpha_\theta = \frac{d^2\mathbf{r}}{d\tau^2} \bullet \frac{\mathbf{rd\theta}}{|\mathbf{rd\theta}|} = -i\frac{M_G}{r^2}(r\sin\theta\frac{d\phi}{d\tau})\frac{dct}{d\tau} .$$

Therefore by the proposition 3. (α_θ) ,we get

$$-i\frac{M_G}{r^2}(r\sin\theta\frac{d\phi}{d\tau})\frac{dct}{d\tau} = 2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta$$

$$= \frac{1}{r}\frac{dr}{d\tau}(r\frac{d\theta}{d\tau}) + \frac{d}{d\tau}(r\frac{d\theta}{d\tau}) - \cos\theta\frac{d\phi}{d\tau}(r\sin\theta\frac{d\phi}{d\tau}) .$$

$(4)_\phi$: The component of $r\sin\theta\mathbf{d\varphi}$- direction is

$$\alpha_\phi = \frac{d^2\mathbf{r}}{d\tau^2} \bullet \frac{\mathbf{rsin\theta d\varphi}}{|\mathbf{rsin\theta d\varphi}|} = i\frac{M_G}{r^2}(r\frac{d\theta}{d\tau})\frac{dct}{d\tau} .$$

Therefore by the proposition 3. (α_ϕ) ,we get

$$i\frac{M_G}{r^2}(r\frac{d\theta}{d\tau})\frac{dct}{d\tau} = 2\dot{r}\dot{\phi}\sin\theta + r\ddot{\phi}\sin\theta + 2r\dot{\phi}\dot{\theta}\cos\theta$$

$$= \frac{1}{r}\frac{dr}{d\tau}(r\sin\theta\frac{d\phi}{d\tau}) + \frac{d}{d\tau}(r\sin\theta\frac{d\phi}{d\tau}) + \cos\theta(r\frac{d\theta}{d\tau})\frac{d\phi}{d\tau} .$$

Q.E.D.

Proposition 3

The acceleration vector at the spherical polar coordinate is

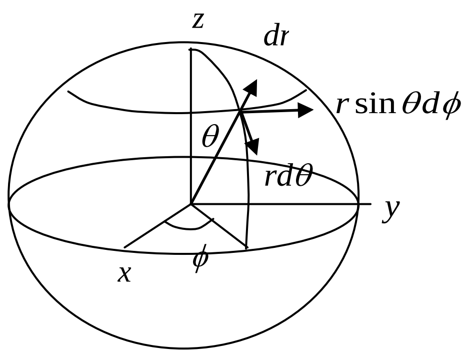
$$\frac{d^2\mathbf{r}}{d\tau^2}=\begin{pmatrix}\alpha_r\\ \alpha_\theta\\ \alpha_\phi\end{pmatrix}=\begin{pmatrix}\ddot{r}-r\dot{\theta}^2-r\dot{\phi}^2\sin^2\theta\\ 2\dot{r}\dot{\theta}+r\ddot{\theta}-r\dot{\phi}^2\sin\theta\cos\theta\\ 2\dot{r}\dot{\phi}\sin\theta+r\ddot{\phi}\sin\theta+2r\dot{\phi}\dot{\theta}\cos\theta\end{pmatrix}.$$

Proof

We use the spherical polar coordinate (r,θ,ϕ).

Let’s $\theta=\theta(\tau)$ and $\phi=\phi(\tau)$ (the function of proper time τ)
be two angles as a right figure.

Then the spherical polar coordinate (r,θ,ϕ) is



$$\begin{cases}x=r\sin\theta\cos\phi\\ y=r\sin\theta\sin\phi\,.\\ z=r\cos\theta\end{cases}$$

And we can represent the position vector as follows;

$$\mathbf{r}=\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix}\begin{pmatrix}0\\ 0\\ r\end{pmatrix}.$$

And let’s $\dot{\theta}(\tau)=\dfrac{d\theta}{d\tau}$ and $\dot{\phi}(\tau)=\dfrac{d\phi}{d\tau}$ be derivatives by the parameter τ (proper time).

Then we can represent the velocity vector as follows;

$$\dot{\mathbf{r}}=\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix}=\begin{pmatrix}-\sin\phi & -\cos\phi & 0\\ \cos\phi & -\sin\phi & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix}\begin{pmatrix}0\\ 0\\ r\dot{\phi}\end{pmatrix}$$

$$+\begin{pmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}-\sin\theta & 0 & \cos\theta\\ 0 & 0 & 0\\ -\cos\theta & 0 & -\sin\theta\end{pmatrix}\begin{pmatrix}0\\ 0\\ r\dot{\theta}\end{pmatrix}$$

$$+\begin{pmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix}\begin{pmatrix}0\\ 0\\ \dot{r}\end{pmatrix}$$

$$=\begin{pmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix}\left(\begin{pmatrix}0\\ r\dot{\phi}\sin\theta\\ 0\end{pmatrix}+\begin{pmatrix}r\dot{\theta}\\ 0\\ 0\end{pmatrix}+\begin{pmatrix}0\\ 0\\ \dot{r}\end{pmatrix}\right)$$

$$=\begin{pmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix}\begin{pmatrix}r\dot{\theta}\\ r\dot{\phi}\sin\theta\\ \dot{r}\end{pmatrix}.$$

For this calculation, we used the following relation;

$$\begin{pmatrix}-\sin\phi & -\cos\phi & 0\\ \cos\phi & -\sin\phi & 0\\ 0 & 0 & 0\end{pmatrix}=\begin{pmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

$$\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix}=\begin{pmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix}\begin{pmatrix}0 & -\cos\theta & 0\\ \cos\theta & 0 & \sin\theta\\ 0 & -\sin\theta & 0\end{pmatrix}.$$

And

$$\begin{pmatrix}-\sin\theta & 0 & \cos\theta\\ 0 & 0 & 0\\ -\cos\theta & 0 & -\sin\theta\end{pmatrix}=\begin{pmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix}\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix}$$

Therefore

$$\begin{pmatrix}v_\theta\\ v_\phi\\ v_r\end{pmatrix}=\begin{pmatrix}r\dot{\theta}\\ r\sin\theta\dot{\phi}\\ \dot{r}\end{pmatrix}$$

is a velocity vector at the spherical polar coordinate.

And let's $\ddot{\theta}(\tau)=\dfrac{d^2\theta}{d\tau^2}$ and $\ddot{\phi}(\tau)=\dfrac{d^2\phi}{d\tau^2}$ be double derivatives by the parameter τ (proper time).

Then we can represent the acceleration vector as follows;

$$\ddot{\mathbf{r}}=\begin{pmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{pmatrix}=\begin{pmatrix}-\sin\phi & -\cos\phi & 0\\ \cos\phi & -\sin\phi & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix}\begin{pmatrix}r\dot{\phi}\dot{\theta}\\ r\dot{\phi}^2\sin\theta\\ \dot{r}\dot{\phi}\end{pmatrix}$$

$$+\begin{pmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}-\sin\theta & 0 & \cos\theta\\ 0 & 0 & 0\\ -\cos\theta & 0 & -\sin\theta\end{pmatrix}\begin{pmatrix}r\dot{\theta}^2\\ r\dot{\phi}\dot{\theta}\sin\theta\\ \dot{r}\dot{\theta}\end{pmatrix}$$

$$+\begin{pmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix}\begin{pmatrix}\dot{r}\dot{\theta}+r\ddot{\theta}\\ \dot{r}\dot{\phi}\sin\theta+r\ddot{\phi}\sin\theta+r\dot{\phi}\dot{\theta}\cos\theta\\ \ddot{r}\end{pmatrix}$$

$$=\begin{pmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix}\begin{pmatrix}0 & -\cos\theta & 0\\ \cos\theta & 0 & \sin\theta\\ 0 & -\sin\theta & 0\end{pmatrix}\begin{pmatrix}r\dot{\phi}\dot{\theta}\\ r\dot{\phi}^2\sin\theta\\ \dot{r}\dot{\phi}\end{pmatrix}$$

$$+\begin{pmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix}\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix}\begin{pmatrix}r\dot{\theta}^2\\ r\dot{\phi}\dot{\theta}\sin\theta\\ \dot{r}\dot{\theta}\end{pmatrix}$$

$$+\begin{pmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix}\begin{pmatrix}\dot{r}\dot{\theta}+r\ddot{\theta}\\ \dot{r}\dot{\phi}\sin\theta+r\ddot{\phi}\sin\theta+r\dot{\phi}\dot{\theta}\cos\theta\\ \ddot{r}\end{pmatrix}$$

$$=\begin{pmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix}\begin{pmatrix}2\dot{r}\dot{\theta}+r\ddot{\theta}-r\dot{\phi}^2\sin\theta\cos\theta\\ 2\dot{r}\dot{\phi}\sin\theta+r\ddot{\phi}\sin\theta+2r\dot{\phi}\dot{\theta}\cos\theta\\ \ddot{r}-r\dot{\theta}^2-r\dot{\phi}^2\sin^2\theta\end{pmatrix}.$$

Therefore

$$\begin{pmatrix}\alpha_\theta\\ \alpha_\phi\\ \alpha_r\end{pmatrix}=\begin{pmatrix}2\dot{r}\dot{\theta}+r\ddot{\theta}-r\dot{\phi}^2\sin\theta\cos\theta\\ 2\dot{r}\dot{\phi}\sin\theta+r\ddot{\phi}\sin\theta+2r\dot{\phi}\dot{\theta}\cos\theta\\ \ddot{r}-r\dot{\theta}^2-r\dot{\phi}^2\sin^2\theta\end{pmatrix}$$

is an acceleration vector at the spherical polar coordinate.

Q.E.D.

References

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