The Equation of Gravitational Force and the Electromagnetic Force (under the relativistic invariant)

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Abstract

In this paper, we discuss the deduction of 4-dimensional equation of motion which is relativistic invariant.

Contents：

In § 1 for preliminaries we mention the modified Maxwell's equation in which we have the time-component of electromagnetic field and use the matrix-vector and relativistic form[1].

In § 2 we consider two forces which is caused by a charge and a mass respectively. These forces are similar in the inverse square law. We improve and push forward the similarity to the potential, field and force.

In § 3 we can deduce the 4-dimensional equation of motion which is relativistic invariant. And in the following paper, this equation contains Kepler's Law and its complex components explain the relativistic effect.

§ 1. Coulomb–Lorentz Force

In the previous paper[1], we can represent Maxwell's equation and its force as a 4-dimensional matrix vector.

Let $\mathbb{E}=\mathbf{E}-ic\mathbf{B}$ be an electric and magnetic field as a complex 3-dimensional field in space and $E_t - icB_t$ $(B_t = 0)$ the time-component.

Then we have a relation of a matrix-vector between 4-dimensional potential $(\phi,\ \mathbf{A})$ and electromagnetic field $(E_t,\ \mathbf{E}-ic\mathbf{B})$ as follows:

$$
{}^{-}\begin{pmatrix} E_t & \\ & \mathbf{E}-ic\mathbf{B} \end{pmatrix}^{+} = {}^{-}\begin{pmatrix} \partial ct & \\ & -\partial \mathbf{r} \end{pmatrix}^{-+} \begin{pmatrix} \phi & \\ & -c\mathbf{A} \end{pmatrix}^{+}
= {}^{-}\begin{pmatrix} \dfrac{\partial \phi}{\partial ct} + div c\mathbf{A} & \\ & -\dfrac{\partial c\mathbf{A}}{\partial ct} - \mathrm{grad}\phi - i\,\mathrm{rot}c\mathbf{A} \end{pmatrix}^{+} . \quad \cdots (*)
$$

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Where signs "+", "−" mean relativistic invariant[1].

We compare the components of this relation, then.

$$
\begin{cases} \underline{E_t} = \dfrac{\partial \phi}{\partial ct} + div\mathbf{A} \cdots (1)' \\ \mathbf{E} = -\dfrac{\partial c\mathbf{A}}{\partial ct} - \mathrm{grad}\phi \cdots (2) \\ c\mathbf{B} = \mathrm{rot}c\mathbf{A} \cdots (3) \end{cases}
$$

Where the above underlined part is a time-component.

And we have a Lorenz gauge $E_t = \dfrac{\partial \phi}{\partial ct} + div c\mathbf{A} = 0$ and a Coulomb gauge $E_t = \dfrac{\partial \phi}{\partial ct}$ $(\Leftrightarrow div c\mathbf{A} = 0)$.

And the Maxwell' s equation is as follows:

$$
{}^{+}\begin{pmatrix} \rho & \\ & -\mathbf{j} \end{pmatrix}^{+} = {}^{+}\begin{pmatrix} \partial ct & \\ & \partial \mathbf{r} \end{pmatrix}^{+-} \begin{pmatrix} E_t & \\ & \mathbf{E}-ic\mathbf{B} \end{pmatrix}^{+}
= {}^{+}\begin{pmatrix} \dfrac{\partial E_t}{\partial ct} + div\,(\mathbf{E}-ic\mathbf{B}) & \\ & \dfrac{\partial\,(\mathbf{E}-ic\mathbf{B})}{\partial ct} + \mathrm{grad}E_t - i\,\mathrm{rot}\,(\mathbf{E}-ic\mathbf{B}) \end{pmatrix}^{+} ,
$$

$$
\begin{cases} \mathrm{rot}\mathbf{E} + \dfrac{\partial c\mathbf{B}}{\partial ct} = 0 \cdots (4) \\ div c\mathbf{B} = 0 \cdots (5) \\ div\,\mathbf{E} + \underline{\dfrac{\partial E_t}{\partial ct}} = \rho \cdots (6)' \\ \mathrm{rot}c\mathbf{B} - \dfrac{\partial \mathbf{E}}{\partial ct} - \underline{\mathrm{grad}E_t} = \mathbf{j} \cdots (7)' \end{cases}
$$

Where the above underlined part is a derivative of time-component.

Therefore the Coulomb-Lorentz force to the moving charge in electromagnetic field is as follows：

$$
{}^{-}\begin{pmatrix} F_t & \\ & \mathbf{F} \end{pmatrix}^{-} = {}^{-}\begin{pmatrix} E_t & \\ & \mathbf{E}-ic\mathbf{B} \end{pmatrix}^{+-} \begin{pmatrix} q & \\ & \mathbf{j} \end{pmatrix}^{-}
= {}^{-}\begin{pmatrix} E_t q + (\mathbf{E}-i\underline{c\mathbf{B}})\cdot\mathbf{j} & \\ & E_t\mathbf{j} + (\mathbf{E}-i\underline{c\mathbf{B}})\cdot q - i\,(\underline{\mathbf{E}}-ic\mathbf{B})\times\mathbf{j} \end{pmatrix}^{-} \cdots (**),
$$

$$
\begin{cases} F_t = qE_t + \mathbf{j}\cdot\mathbf{E} - i\,\underline{\mathbf{j}\cdot c\mathbf{B}} & \text{(the variation of energy)} \\ \mathbf{F} = q\mathbf{E} + \mathbf{j}E_t + \mathbf{j}\times c\mathbf{B} - i\,(\underline{qc\mathbf{B} - \mathbf{j}\times\mathbf{E}}) & \text{(the variation of momentum)} \end{cases} .
$$

Where the above underlined part is a complex force.

§ 2. Coulomb-Lorentz force and gravitational one

We consider the 4-dimensional potential $\phi\,(x,\ y,\ z) = \dfrac{1}{4\pi\varepsilon_0}\dfrac{e}{r}$ (ε_0 is a dielectric constant) and $A(x,\ y,\ z) = 0$ which are caused by the stationary charge "e".

Then the 4-dimensional electromagnetic field $(E_t,\ \mathrm{E}-ic\mathrm{B})$ is given by the above formula (*),

$$
{}^{-}\begin{pmatrix} E_t & \\ & \mathbf{E}-ic\mathbf{B} \end{pmatrix}^{+} = {}^{-}\begin{pmatrix} \partial ct & \\ & -\partial \mathbf{r} \end{pmatrix}^{-+} \begin{pmatrix} \dfrac{1}{4\pi\varepsilon_0}\dfrac{e}{\mathrm{r}} & \\ & \mathbf{0} \end{pmatrix}^{+} = \frac{1}{4\pi\varepsilon_0}\,{}^{-}\begin{pmatrix} 0 & \\ & -\dfrac{\partial}{\partial \mathbf{r}}\left(\dfrac{e}{r}\right) \end{pmatrix}^{+} .
$$

That is, the electric field is

$$\mathbf{E}=-\frac{1}{4\pi\varepsilon_0}\frac{\partial}{\partial \mathbf{r}}\left(\frac{e}{r}\right)=\frac{e}{4\pi\varepsilon_0 r^2}\frac{\mathbf{r}}{r}.$$

And the magnetic field and the time-component are
$\mathbf{B}=\mathbf{0}$ and $E_t=0$.

And we put $(q,\mathbf{j})=(q_0\gamma,\ q_0\gamma\boldsymbol{\beta})=\left(\frac{q_0}{c}u_t,\ \frac{q_0}{c}\mathbf{u}\right)$ where $u_t=\frac{dct}{d\tau}=c\gamma$, $\mathbf{u}=\frac{d\mathbf{r}}{d\tau}=c\gamma\boldsymbol{\beta}$.

Then by the above formula (＊＊), the Coulomb-Lorentz force which acts on the moving charge $(q,\mathbf{j})$ in the electromagnetic field is

$${}^{-}\begin{pmatrix}F_t & \\ & \mathbf{F}\end{pmatrix}^{-}={}^{-}\begin{pmatrix}\partial ct & \\ & -\partial\mathbf{r}\end{pmatrix}^{-+}\begin{pmatrix}\frac{1}{4\pi\varepsilon_0}\frac{e}{\mathrm{r}} & \\ & \mathbf{0}\end{pmatrix}^{+}\frac{q_0}{c}{}^{-}\begin{pmatrix}u_t & \\ & \mathbf{u}\end{pmatrix}^{-}$$

$$=\frac{1}{4\pi\varepsilon_0}{}^{-}\begin{pmatrix}0 & \\ & -\frac{\partial}{\partial\mathbf{r}}\left(\frac{e}{r}\right)\end{pmatrix}^{+}\frac{q_0}{c}{}^{-}\begin{pmatrix}u_t & \\ & \mathbf{u}\end{pmatrix}^{-}$$

$$=\frac{q_0}{4\pi\varepsilon_0 c}{}^{-}\begin{pmatrix}-\frac{\partial}{\partial\mathbf{r}}\left(\frac{e}{r}\right)\cdot\mathbf{u} & \\ & -\frac{\partial}{\partial\mathbf{r}}\left(\frac{e}{r}\right)u_t+i\frac{\partial}{\partial\mathbf{r}}\left(\frac{e}{r}\right)\times\mathbf{u}\end{pmatrix}^{-}\cdots(＊＊＊),$$

$$\begin{cases}F_t=-\frac{q_0}{4\pi\varepsilon_0 c}\frac{\partial}{\partial\mathbf{r}}\left(\frac{e}{r}\right)\cdot\mathbf{u} & \text{(the variation of energy)}\\ \mathbf{F}=-\frac{q_0}{4\pi\varepsilon_0 c}\frac{\partial}{\partial\mathbf{r}}\left(\frac{e}{r}\right)u_t+i\underline{\frac{q_0}{4\pi\varepsilon_0 c}\frac{\partial}{\partial\mathbf{r}}\left(\frac{e}{r}\right)\times\mathbf{u}} & \text{(the variation of momentum).}\end{cases}$$

The above underlined part is a complex force.

We compare this force and the gravitational one which is caused by the stationary mass "M" (for simplicity) as follows：

The relation of its potential $U=\frac{G}{c^2}\frac{M}{r}$ and gravitational force $\mathbf{f}$ is

$$\mathbf{f}=-m_0\frac{\partial U}{\partial\mathbf{r}}\fallingdotseq-m_0\underline{\gamma}\frac{\partial U}{\partial\mathbf{r}}=-\frac{Gm_0}{c^2}\frac{\partial}{\partial\mathbf{r}}\left(\frac{M}{r}\right)u_t.$$

Where $\underline{\gamma}=\frac{u_t}{c}=\frac{dct}{c\cdot d\tau}=\frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}(\fallingdotseq 1)$, G is a gravitational constant and c is a light velocity.

This gravitational force $\mathbf{f}$ is quite similar to the real part of the Coulomb-Lorentz one

$$\mathbf{F}=-\frac{q_0}{4\pi\varepsilon_0 c}\frac{\partial}{\partial\mathbf{r}}\left(\frac{e}{r}\right)u_t+i\ \text{(imaginary part)}.$$

Therefore, we get the 4-dimentional force $(f_t,\mathbf{f})$ which is caused by the stationary mass "M", that is, the potential is

$${}^{+}\begin{pmatrix}U & \\ & \mathbf{0}\end{pmatrix}^{+}={}^{+}\begin{pmatrix}\frac{G}{c^2}\frac{M}{r} & \\ & \mathbf{0}\end{pmatrix}^{+}\text{ which is corresponding to }{}^{+}\begin{pmatrix}\frac{1}{4\pi\varepsilon_0}\frac{e}{r} & \\ & \mathbf{0}\end{pmatrix}^{+}.$$

And its gravitational field is

$${}^{-}\begin{pmatrix}\partial ct & \\ & -\partial\mathbf{r}\end{pmatrix}^{-+}\begin{pmatrix}\frac{G}{c^2}\frac{M}{r} & \\ & \mathbf{0}\end{pmatrix}^{+}=\frac{G}{c^2}{}^{-}\begin{pmatrix}0 & \\ & -\frac{\partial}{\partial\mathbf{r}}\left(\frac{M}{r}\right)\end{pmatrix}^{+}.$$

Therefore we get the 4-dimentional gravitational force as follows；

$${}^{-}\begin{pmatrix}f_t & \\ & \mathbf{f}\end{pmatrix}^{-}={}^{-}\begin{pmatrix}\partial ct & \\ & -\partial\mathbf{r}\end{pmatrix}^{-+}\begin{pmatrix}\frac{G}{c^2}\frac{M}{r} & \\ & \mathbf{0}\end{pmatrix}^{+}\frac{m_0}{c}{}^{-}\begin{pmatrix}u_t & \\ & \mathbf{u}\end{pmatrix}^{-}$$

$$=\frac{G}{c^2}{}^{-}\begin{pmatrix}0 & \\ & -\frac{\partial}{\partial\mathbf{r}}\left(\frac{M}{r}\right)\end{pmatrix}^{+}\frac{m_0}{c}{}^{-}\begin{pmatrix}u_t & \\ & \mathbf{u}\end{pmatrix}^{-}$$

$$=\frac{Gm_0}{c^3}{}^{-}\begin{pmatrix}-\frac{\partial}{\partial\mathbf{r}}\left(\frac{M}{r}\right)\cdot\mathbf{u} & \\ & -\frac{\partial}{\partial\mathbf{r}}\left(\frac{M}{r}\right)\mathrm{u}_t+i\frac{\partial}{\partial\mathbf{r}}\left(\frac{M}{r}\right)\times\mathbf{u}\end{pmatrix}^{-}.$$

That is,

$$\begin{cases}f_t=-\frac{Gm_0}{c^3}\frac{\partial}{\partial\mathbf{r}}\left(\frac{M}{r}\right)\cdot\mathbf{u} & \text{(the variation of energy)}\\ \mathbf{f}=-\frac{Gm_0}{c^3}\frac{\partial}{\partial\mathbf{r}}\left(\frac{M}{r}\right)u_t+i\underline{\frac{Gm_0}{c^3}\frac{\partial}{\partial\mathbf{r}}\left(\frac{M}{r}\right)\times\mathbf{u}} & \text{(the variation of momentum).}\end{cases}$$

The above underlined part is a complex force and its interpretation is in the following paprer.

§3. The 4-dimensional equation of motion which is relativistic invariant

In the above discussion, we had correspond the source charge e to the source mass M, the moving charge $(q,\mathbf{j})=(q_0\gamma,\ q_0\gamma\boldsymbol{\beta})$ to the moving mass $(m_0\gamma,\ m_0\gamma\boldsymbol{\beta})$ and the constant $\frac{1}{4\pi\varepsilon_0}$ of the Coulomb-Lorentz force to the gravitational constant $\frac{\mathrm{G}}{c^2}$. Then we get the modified equation of motion.

> **Theorem 1**
>
> The equation of motion which is relativistic invariant is
>
> $$\begin{cases}\frac{d^2ct}{d\tau^2}=-\frac{M_G}{r^2}\left(\frac{\mathbf{r}}{r}\cdot\frac{d\mathbf{r}}{d\tau}\right)\frac{dct}{d\tau}\cdots(1)_{ct}\\ \frac{d^2\mathbf{r}}{d\tau^2}=-\frac{M_G}{r^2}\frac{\mathbf{r}}{r}\left(\frac{dct}{d\tau}\right)^2+i\frac{M_G}{r^2}\left(\frac{\mathbf{r}}{r}\times\frac{d\mathbf{r}}{d\tau}\right)\frac{dct}{d\tau}\cdots(2)_r+i\{(3)_\theta+(4)_\phi\}.\end{cases}$$

Proof

We replace $\frac{Q}{r}=\frac{1}{4\pi\varepsilon_0}\frac{e}{r}$ (potential of "negative" stationary charge), $(q,\mathbf{j})=(q_0\gamma,\ q_0\gamma\boldsymbol{\beta})$ which is "positive moving charge" as $\frac{M_G}{r}=\frac{G}{c^2}\frac{M}{r}$ (potential of stationary mass), $(m_0\gamma,\ m_0\gamma\boldsymbol{\beta})$ which is "moving mass" in the formula (＊＊＊).

And by this replacement, we get the 4-dimensional gravitational force as follows：

$${}^{-}\begin{pmatrix}f_t & \\ & \mathbf{f}\end{pmatrix}^{-}=\underline{{}^{-}\begin{pmatrix}\partial ct & \\ & -\partial\mathbf{r}\end{pmatrix}^{-+}\begin{pmatrix}\frac{M_G}{r} & \\ & 0\end{pmatrix}^{+}}\frac{m_0}{c}{}^{-}\begin{pmatrix}u_t & \\ & \mathbf{u}\end{pmatrix}^{-},\quad M_G=\frac{GM}{c^2}.$$

Where the underlined part is a 4-dimensional gravitational field.

And we integrate this formula by time then

$$\int_{t_0}^{t}{}^{-}\begin{pmatrix}f_t & \\ & \mathbf{f}\end{pmatrix}^{-}cdt=\int_{t_0}^{t}{}^{-}\begin{pmatrix}\partial ct & \\ & -\partial\mathbf{r}\end{pmatrix}^{-+}\begin{pmatrix}\frac{M_G}{r} & \\ & 0\end{pmatrix}^{+}\frac{m_0}{c}{}^{-}\begin{pmatrix}u_t & \\ & \mathbf{u}\end{pmatrix}^{-}cdt$$

means a variation of energy-momentum

$$\left[{}^{-}\begin{pmatrix} m_0\gamma & \\ & m_0\gamma\boldsymbol{\beta} \end{pmatrix}^{-}\right]_{t_0}^{t}.$$

Therefore we get the modified equation of motion as follows ;

$$m_0\frac{d}{d\tau}\,{}^{-}\begin{pmatrix} \frac{dct}{d\tau} & \\ & \frac{d\mathbf{r}}{d\tau} \end{pmatrix}^{-} = \mathrm{c}\frac{\mathrm{d}}{\mathrm{d}\tau}\,{}^{-}\begin{pmatrix} m_0\gamma & \\ & m_0\gamma\boldsymbol{\beta} \end{pmatrix}^{-}$$

$$= M_G m_0\,{}^{-}\begin{pmatrix} \partial ct & \\ & -\partial\mathbf{r} \end{pmatrix}^{-+}\begin{pmatrix} \frac{1}{r} & \\ & \mathbf{0} \end{pmatrix}^{+}\,{}^{-}\begin{pmatrix} \frac{dct}{d\tau} & \\ & \frac{d\mathbf{r}}{d\tau} \end{pmatrix}^{-}\frac{dct}{d\tau}$$

$$= -\frac{M_G m_0}{r^2}\,{}^{-}\begin{pmatrix} \left(\frac{\mathbf{r}}{r}\cdot\frac{d\mathbf{r}}{d\tau}\right)\left(\frac{dct}{d\tau}\right) & \\ & \frac{\mathbf{r}}{r}\left(\frac{dct}{d\tau}\right)^2 - i\left(\frac{\mathbf{r}}{r}\times\frac{d\mathbf{r}}{d\tau}\right)\left(\frac{dct}{d\tau}\right) \end{pmatrix}^{-}.$$

Q.E.D.

We can rewrite the coordinate (x, y, z) by the spherical polar coordinate (r, θ, ϕ), that is,

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}.$$

Then we get

Corollary 2

The equation of motion at the spherical polar coordinate is

$$\begin{cases} \frac{d^2ct}{d\tau^2} = -\frac{M_G}{r^2}\frac{dr}{d\tau}\frac{dct}{d\tau}\cdots(1)_{ct} \\ \frac{d^2r}{d\tau^2} = -\frac{M_G}{r^2}\left(\frac{dct}{d\tau}\right)^2 + \frac{1}{r}\left(r\frac{d\theta}{d\tau}\right)^2 + \frac{1}{r}\left(r\sin\theta\frac{d\phi}{d\tau}\right)^2\cdots(2)_r \\ \frac{d}{d\tau}\left(r\frac{d\theta}{d\tau}\right) = -i\frac{M_G}{r^2}\left(r\sin\theta\frac{d\phi}{d\tau}\right)\frac{dct}{d\tau} - \frac{1}{r}\frac{dr}{d\tau}\left(r\frac{d\theta}{d\tau}\right) + \cos\theta\frac{d\phi}{d\tau}\left(r\sin\theta\frac{d\phi}{d\tau}\right)\cdots i(3)_\theta \\ \frac{d}{d\tau}\left(r\sin\theta\frac{d\phi}{d\tau}\right) = i\frac{M_G}{r^2}\left(r\frac{d\theta}{d\tau}\right)\frac{dct}{d\tau} - \frac{1}{r}\frac{dr}{d\tau}\left(r\sin\theta\frac{d\phi}{d\tau}\right) - \cos\theta\left(r\frac{d\theta}{d\tau}\right)\frac{d\phi}{d\tau}\cdots i(4)_\phi \end{cases}.$$

Proof :

$(1)_{ct}$: Formula $(1)_{ct}$ is the same one

And by the proposition 3 below, we get the formulas $(2)_r$, $(3)_\theta$, $(4)_\phi$ as follows :

By the theorem 1

$$\frac{d^2\mathbf{r}}{d\tau^2} = -\frac{M_G}{r^2}\frac{\mathbf{r}}{r}\left(\frac{dct}{d\tau}\right)^2 + i\frac{M_G}{r^2}\left(\frac{\mathbf{r}}{r}\times\frac{d\mathbf{r}}{d\tau}\right)\frac{dct}{d\tau}\cdots(2)_r + i\{(3)_\theta + (4)_\phi\},$$

$$\frac{d\mathbf{r}}{d\tau} = \begin{pmatrix} \nu_r \\ \nu_\theta \\ \nu_\phi \end{pmatrix} = \begin{pmatrix} \dot{r} \\ r\dot{\theta} \\ r\sin\theta\dot{\phi} \end{pmatrix} \text{ and } \frac{\mathbf{r}}{r}\times\frac{d\mathbf{r}}{d\tau} = \begin{pmatrix} 0 \\ -r\sin\theta\dot{\phi} \\ r\dot{\theta} \end{pmatrix}.$$

$(2)_r$: The component of r-direction is

$$a_r = \frac{d^2\mathbf{r}}{d\tau^2}\cdot\frac{\mathbf{r}}{|r|} = -\frac{M_G}{r^2}\left(\frac{dct}{d\tau}\right)^2 \quad \text{(where " $\cdot$ " is an inner product).}$$

Therefore

$$-\frac{M_G}{r^2}\left(\frac{dct}{d\tau}\right)^2 = \ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta$$

$$= \frac{d^2r}{d\tau^2} - \frac{1}{r}\left(r\frac{d\theta}{d\tau}\right)^2 - \frac{1}{r}\left(r\sin\theta\frac{d\phi}{d\tau}\right)^2.$$

$(3)_\theta$: The component of $rd\theta$-direction is

$$a_\theta = \frac{d^2\mathbf{r}}{d\tau^2}\cdot\frac{\boldsymbol{rd\theta}}{|rd\theta|} = -i\frac{M_G}{r^2}\left(r\sin\theta\frac{d\phi}{d\tau}\right)\frac{dct}{d\tau}.$$

Therefore

$$-i\frac{M_G}{r^2}\left(r\sin\theta\frac{d\phi}{d\tau}\right)\frac{dct}{d\tau} = 2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta$$

$$= \frac{1}{r}\frac{dr}{d\tau}\left(r\frac{d\theta}{d\tau}\right) + \frac{d}{d\tau}\left(r\frac{d\theta}{d\tau}\right) - \cos\theta\frac{d\phi}{d\tau}\left(r\sin\theta\frac{d\phi}{d\tau}\right).$$

$(4)_\phi$: The component of $r\sin\theta d\phi$-direction is

$$a_\phi = \frac{d^2\mathbf{r}}{d\tau^2}\cdot\frac{\boldsymbol{r\sin\theta d\phi}}{|r\sin\theta d\phi|} = i\frac{M_G}{r^2}\left(r\frac{d\theta}{d\tau}\right)\frac{dct}{d\tau}.$$

Therefore

$$i\frac{M_G}{r^2}\left(r\frac{d\theta}{d\tau}\right)\frac{dct}{d\tau} = 2\dot{r}\dot{\phi}\sin\theta + r\ddot{\phi}\sin\theta + 2r\dot{\phi}\dot{\theta}\cos\theta$$

$$= \frac{1}{r}\frac{dr}{d\tau}\left(r\sin\theta\frac{d\phi}{d\tau}\right) + \frac{d}{d\tau}\left(r\sin\theta\frac{d\phi}{d\tau}\right) + \cos\theta\left(r\frac{d\theta}{d\tau}\right)\frac{d\phi}{d\tau}.$$

Q.E.D.

Proposition 3

The acceleration vector at the spherical polar coordinate is

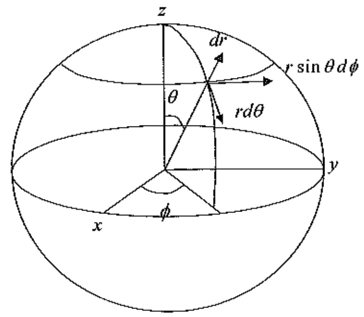
$$\frac{d^2\mathrm{r}}{d\tau^2} = \begin{pmatrix} a_r \\ a_\theta \\ a_\phi \end{pmatrix} = \begin{pmatrix} \ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta \\ 2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta \\ 2\dot{r}\dot{\phi}\sin\theta + r\ddot{\phi}\sin\theta + 2r\dot{\phi}\dot{\theta}\cos\theta \end{pmatrix}.$$

Proof

We use the spherical polar coordinate (r, θ, ϕ).

Let's $\theta = \theta(\tau)$ and $\phi = \phi(\tau)$ (the function of proper time) be two angles as a right figure. Then the spherical polar coordinate (r, θ, ϕ) is

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$



And we can represent the position vector as follows;

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix}.$$

And let's $\dot{\theta}(\tau) = \frac{d\theta}{d\tau}$ and $\dot{\phi}(\tau) = \frac{d\phi}{d\tau}$ be derivatives by the parameter τ (proper time).

Then we can represent the velocity vector as follows ;

$$\dot{\mathbf{r}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -\sin\phi & -\cos\phi & 0 \\ \cos\phi & -\sin\phi & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ r\dot{\phi} \end{pmatrix}$$

$$+ \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\sin\theta & 0 & \cos\theta \\ 0 & 0 & 0 \\ -\cos\theta & 0 & -\sin\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ r\dot{\theta} \end{pmatrix}$$

$$+ \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot{r} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \left(\begin{pmatrix} 0 \\ r\dot{\phi}\sin\theta \\ 0 \end{pmatrix} + \begin{pmatrix} r\dot{\theta} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{r} \end{pmatrix} \right)$$

$$= \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} r\dot{\theta} \\ r\dot{\phi}\sin\theta \\ \dot{r} \end{pmatrix}.$$

For this calculation, we used the following relations

$$\begin{pmatrix} -\sin\phi & -\cos\phi & 0 \\ \cos\phi & -\sin\phi & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} -\sin\theta & 0 & \cos\theta \\ 0 & 0 & 0 \\ -\cos\theta & 0 & -\sin\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

And

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 0 & -\cos\theta & 0 \\ \cos\theta & 0 & \sin\theta \\ 0 & -\sin\theta & 0 \end{pmatrix}.$$

Therefore

$$\begin{pmatrix} \nu_\theta \\ \nu_\phi \\ \nu_r \end{pmatrix} = \begin{pmatrix} r\dot{\theta} \\ r\sin\theta\dot{\phi} \\ \dot{r} \end{pmatrix}$$

is a velocity vector at the spherical polar coordinate.

And let's $\ddot{\theta}(\tau) = \frac{d^2\theta}{d\tau^2}$ and $\ddot{\phi}(\tau) = \frac{d^2\phi}{d\tau^2}$ be double derivatives by parameter τ.

Then we can represent the acceleration vector as follows ;

$$\ddot{\mathbf{r}} = \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = \begin{pmatrix} -\sin\phi & -\cos\phi & 0 \\ \cos\phi & -\sin\phi & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} r\dot{\phi}\dot{\theta} \\ r\dot{\phi}^2\sin\theta \\ \dot{r}\dot{\phi} \end{pmatrix}$$

$$+ \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\sin\theta & 0 & \cos\theta \\ 0 & 0 & 0 \\ -\cos\theta & 0 & -\sin\theta \end{pmatrix} \begin{pmatrix} r\dot{\theta}^2 \\ r\dot{\phi}\dot{\theta}\sin\theta \\ \dot{r}\dot{\theta} \end{pmatrix}$$

$$+ \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \dot{r}\dot{\theta} + r\ddot{\theta} \\ \dot{r}\dot{\phi}\sin\theta + r\ddot{\phi}\sin\theta + r\dot{\phi}\dot{\theta}\cos\theta \\ \ddot{r} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 0 & -\cos\theta & 0 \\ \cos\theta & 0 & \sin\theta \\ 0 & -\sin\theta 0 & 0 \end{pmatrix} \begin{pmatrix} r\dot{\phi}\dot{\theta} \\ r\dot{\phi}^2\sin\theta \\ \dot{r}\dot{\phi} \end{pmatrix}$$

$$+ \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} r\dot{\theta}^2 \\ r\dot{\phi}\dot{\theta}\sin\theta \\ \dot{r}\dot{\theta} \end{pmatrix}$$

$$+ \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \dot{r}\dot{\theta} + r\ddot{\theta} \\ \dot{r}\dot{\phi}\sin\theta + r\ddot{\phi}\sin\theta + r\dot{\phi}\dot{\theta}\cos\theta \\ \ddot{r} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta \\ 2\dot{r}\dot{\phi}\sin\theta + r\ddot{\phi}\sin\theta + 2r\dot{\phi}\dot{\theta}\cos\theta \\ \ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta \end{pmatrix}.$$

Therefore

$$\begin{pmatrix} \alpha_\theta \\ \alpha_\phi \\ \alpha_r \end{pmatrix} = \begin{pmatrix} 2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\phi}^2 \sin\theta\cos\theta \\ 2\dot{r}\dot{\phi}\sin\theta + r\ddot{\phi}\sin\theta + 2r\dot{\phi}\dot{\theta}\cos\theta \\ \ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta \end{pmatrix}$$

is an acceleration vector at the spherical polar coordinate.
Q.E.D.

References

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