~~運動する電荷の運動方程式(行列表記)とラグランジアン~~

A New Form of Equation of Motion for a Moving Charge

and

the Lagrangian

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In our previous paper, we presented a new notion, "matrix-vector", which is a vector where the function of matrix product has been added [(8) Y. Takemoto, Bull. of NBU Vol.34, No.1 (2006-Mar.) p.32].

In this paper, as an application of the matrix-vectors, we deduce an equation of motion represented by matrix for a moving charge in an electromagnetic field.

Contents:

In §1, using a traditional variational method, we deduce (A)the usual 4-dimensional momentum and (B)equation of motion from the Lagrangian. Now we rewrite its momentum and equation into the matrix-vector form.

In §2, for preliminaries, we review (A)the matrix-vector and (B)its Lorentz form. Now we define the variation of the matrix-vector and investigate its meaning by comparing this variation with the usual one ($\delta\phi$, $\delta\mathbf{A}$).

In §3, we denote the Lagrangian by the matrix-vector form and use the variational method. Then we can get the equation of motion which is represented by matrix-vector form.

New features of this equation are

(1) New effects of the time components E_0 of the electric field appear.

(2) The 4-dimensional complex force appears.

(3) The relativistic invariance of the equation is apparent.

§1. Introduction

We put the Lagrangians L and L_0 which are for time dt and for proper time $ds=\sqrt{(dct)^2-(d\mathbf{r})^2}$ respectively, that is,

$$L=-mc^2\sqrt{1-\frac{\mathbf{v}^2}{c^2}}+q(\mathbf{A}\cdot\mathbf{v}-\phi)\,,$$

$$L_0=-mc+\frac{q}{c}(c\mathbf{A}\cdot\frac{\mathbf{u}}{c}-\phi\frac{u_0}{c})\,,$$

where $\dfrac{u_0}{c}=\dfrac{dct}{ds}=\dfrac{1}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}=\gamma\,,\quad \dfrac{\mathbf{u}}{c}=\dfrac{d\mathbf{r}}{ds}=\dfrac{\frac{\mathbf{v}}{c}}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}=\gamma\boldsymbol{\beta}\,.$

The actions of these Lagrangians are as follows:

$$S=\int_a^b\{-mc+\frac{q}{c}(c\mathbf{A}\cdot\frac{\mathbf{u}}{c}-\phi\frac{u_0}{c})\}ds=\int_{t_1}^{t_2}\{-mc^2\sqrt{1-\frac{v^2}{c^2}}+q(\mathbf{A}\cdot\mathbf{v}-\phi)\}dt\,.$$

Further we put the variations.

$$\delta\phi=\frac{d\phi}{dct}\delta ct+\mathbf{grad}\phi\cdot\delta\mathbf{r}\,,\quad \delta\mathbf{A}=\frac{d\mathbf{A}}{dct}\delta ct+div\mathbf{A}\delta\mathbf{r}\cdots\cdots(*)\,.$$

And we use the relation ${u_0}^2-\mathbf{u}^2=c^2$, then

$$\delta ds=u_0d(\delta ct)-\mathbf{u}d(\delta\mathbf{r})\,,\quad \delta(u_0)u_0-\delta(\mathbf{u})\mathbf{u}=0\cdots\cdots\cdots(**)\,.$$

We get (A)the usual generalized momentum and (B)equation of motion to the moving charge q with the mass m in the electromagnetic field as follows:

(A) The generalized momentum is

$$\mathbf{P}=\frac{\partial L}{\partial\mathbf{v}}=\frac{m\mathbf{v}}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}+q\mathbf{A}=\mathbf{p}+q\mathbf{A}\,.$$

The generalized energy is

$$\mathrm{E}=\mathbf{P}\cdot\mathbf{v}-L=\frac{mc^2}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}+q\phi=\varepsilon+q\phi\,.$$

And the relation between them is

$$(\mathrm{E}-q\phi)^2-(\mathbf{P}c-q\mathbf{A})^2=\varepsilon^2-(\mathbf{p}c)^2=(mc^2)^2 .$$

(B) The equation of motion is

$$\frac{d}{dt}(\frac{\partial L}{\partial \mathbf{v}})=\frac{\partial L}{\partial \mathbf{r}} .$$

The left side term is

$$\frac{d}{dt}(\frac{\partial L}{\partial \mathbf{v}})=\frac{\partial \mathbf{P}}{\partial t}=\frac{d\mathbf{p}}{dt}+\frac{q}{c}(\frac{\partial \mathbf{A}}{\partial t}+(\mathbf{v}\cdot\mathbf{grad})\mathbf{A}) .$$

The right side term is

$$\frac{\partial L}{\partial \mathbf{r}}=\mathbf{grad}L=q\mathbf{grad}(\mathbf{A}\cdot\mathbf{v})-q\mathbf{grad}\phi=q((\mathbf{v}\cdot\mathbf{grad})\mathbf{A}+\mathbf{v}\times\mathbf{rotA})-q\mathbf{grad}\phi \cdot$$

Therefore we get the equation of motion of moving charge in the electromagnetic field.

$$\frac{d\mathbf{p}}{dt}=-q(\frac{\partial \mathbf{A}}{\partial t}+\mathbf{grad}\phi)+q\mathbf{v}\times\mathbf{rotA} ,$$

$$=q\mathbf{E}+q\frac{\mathbf{v}}{c}\times c\mathbf{B}-iq\underline{[c\mathbf{B}-\frac{\mathbf{v}}{c}\times\mathbf{E}]} .$$

where $\mathbf{E}=-\frac{\partial \mathbf{A}}{\partial t}-\mathbf{grad}\phi ,\quad \mathbf{B}=\mathbf{rotA}$.

The **underlined imaginary part** is the term which we have added.

Using $\varepsilon^2-(\mathbf{p}c)^2=(mc^2)^2$ and $\mathbf{p}=\frac{\varepsilon\mathbf{v}}{c^2}$, we get

$$\frac{d\varepsilon}{dt}=\mathbf{v}\cdot\frac{d\mathbf{p}}{dt} ,$$

$$=\mathbf{v}\cdot(q\mathbf{E}+q\frac{\mathbf{v}}{c}\times c\mathbf{B})-\mathbf{v}\cdot iq\underline{[c\mathbf{B}-\frac{\mathbf{v}}{c}\times\mathbf{E}]} ,$$

$$=q\mathbf{v}\cdot\mathbf{E}-i\underline{q\mathbf{v}\cdot c\mathbf{B}} .$$

We can rewrite these equations by using the matrix-vector as follows:

$$\frac{d}{dt}\begin{pmatrix}\frac{\varepsilon}{c} & \\ & \mathbf{p}\end{pmatrix}=\begin{pmatrix}\frac{q}{c}\mathbf{v}\cdot\mathbf{E}-i\underline{\frac{q}{c}\mathbf{v}\cdot c\mathbf{B}} & \\ & q\mathbf{E}+q\frac{\mathbf{v}}{c}\times c\mathbf{B}-iq\underline{[c\mathbf{B}-\frac{\mathbf{v}}{c}\times\mathbf{E}]}\end{pmatrix} ,$$

$$=\frac{q}{c}\begin{pmatrix}0 & \\ & \mathbf{E}-ic\mathbf{B}\end{pmatrix}\begin{pmatrix}c & \\ & \mathbf{v}\end{pmatrix} .$$

§2. Preliminaries and notations.

In this section, we review (I)-(A)a **matrix-vector** and (B)its **Lorentz form**. Now we define (II)the variation of the matrix-vector.

(I)-(A) A matrix-vector. [6)8)]

We **identify** the 4-dimensional vector $\begin{pmatrix} A_t \\ \\ \mathbf{A} \\ \\ \end{pmatrix} = \begin{pmatrix} A_t \\ A_x \\ A_y \\ A_z \end{pmatrix} \in \mathbb{R}^4$ and the $u(1)-$ matrix[2)3)]

$\begin{pmatrix} A_t + A_x & A_y + iA_z \\ A_y - iA_z & A_t - A_x \end{pmatrix}$, and we represent this matrix by a symbol

$\begin{pmatrix} A_t & \\ & \mathbf{A} \end{pmatrix} = \begin{pmatrix} A_t & \\ & (A_x \quad A_y \quad A_z) \end{pmatrix}$ and call it a **matrix-vector**.

And we **complexify** the each component A_t, A_x, A_y, A_z, that is, we define the symbol

$\begin{pmatrix} A_t & \\ & \mathbf{A} \end{pmatrix} = \begin{pmatrix} A_t & \\ & (A_x \quad A_y \quad A_z) \end{pmatrix}$ as the matrix $\begin{pmatrix} A_t + A_x & A_y + iA_z \\ A_y - iA_z & A_t - A_x \end{pmatrix}$ with complex components.

Then the product(4-dimensional **vector product**) between two matrix-vector is as follows:

$$\begin{pmatrix} A_t & \\ & \mathbf{A} \end{pmatrix} \begin{pmatrix} B_t & \\ & \mathbf{B} \end{pmatrix} = \begin{pmatrix} A_t B_t + \underline{\mathbf{A} \cdot \mathbf{B}} & \\ & A_t \mathbf{B} + \mathbf{A} B_t - i(\underline{\underline{\mathbf{A} \times \mathbf{B}}}) \end{pmatrix}.$$

And we define $\begin{pmatrix} A_t & \\ & \mathbf{A} \end{pmatrix}_{\mathbf{T}}$, $\begin{pmatrix} A_t & \\ & \mathbf{A} \end{pmatrix}_{\mathbf{S}}$ and $\overline{\begin{pmatrix} A_t & \\ & \mathbf{A} \end{pmatrix}}$ are each the time part, space part and a conjugate[4)] of $\begin{pmatrix} A_t & \\ & \mathbf{A} \end{pmatrix}$ respectively.

This conjugate corresponds to the cofactor matrix of matrix $\begin{pmatrix} A_t + A_x & A_y + iA_z \\ A_y - iA_z & A_t - A_x \end{pmatrix}$.

Therefore we get the relation:

$$\begin{pmatrix} A_t & \\ & \mathbf{A} \end{pmatrix}\begin{pmatrix} B_t & \\ & \mathbf{B} \end{pmatrix} = \begin{pmatrix} B_t & \\ & \mathbf{B} \end{pmatrix}\begin{pmatrix} A_t & \\ & \mathbf{A} \end{pmatrix}, \quad \text{and}$$

$$\left[\begin{pmatrix} A_t & \\ & \mathbf{A} \end{pmatrix}\begin{pmatrix} B_t & \\ & \mathbf{B} \end{pmatrix}\right]_{\mathbf{T}} = \left[\begin{pmatrix} B_t & \\ & \mathbf{B} \end{pmatrix}\begin{pmatrix} A_t & \\ & \mathbf{A} \end{pmatrix}\right]_{\mathbf{T}} \cdots\cdots\cdots (***).$$

(B) The Lorentz form. [8)]

When a particle moves to the x-direction at the speed v, then we have the **Lorentz transformation**:

$$\begin{cases} ct' = \gamma(ct - \beta x) \\ x' = \gamma(x - \beta ct) \\ y' = y \\ z' = z \end{cases} .$$

where $\gamma = \dfrac{1}{\sqrt{1-(\frac{v}{c})^2}} = \cosh\Theta$ and $\gamma\beta = \dfrac{\frac{v}{c}}{\sqrt{1-(\frac{v}{c})^2}} = \sinh\Theta$.

And we can rewrite this **transformation** by using the matrix:

$$\begin{pmatrix} ct'+x' & y'+iz' \\ y'-iz' & ct'-x' \end{pmatrix} = \begin{pmatrix} \gamma(1-\beta)(ct+x) & y+iz \\ y-iz & \gamma(1+\beta)(ct-x) \end{pmatrix},$$

$$= \begin{pmatrix} \gamma_+ - \gamma_- & 0 \\ 0 & \gamma_+ + \gamma_- \end{pmatrix}\begin{pmatrix} ct+x & y+iz \\ y-iz & ct-x \end{pmatrix}\begin{pmatrix} \gamma_+ - \gamma_- & 0 \\ 0 & \gamma_+ + \gamma_- \end{pmatrix},$$

where $\gamma_+ = \sqrt{\dfrac{\gamma+1}{2}} = \cosh\dfrac{\Theta}{2}$ and $\gamma_- = \sqrt{\dfrac{\gamma-1}{2}} = \sinh\dfrac{\Theta}{2}$.

Then we have a **relativistic** transformation in the **matrix-vector** form:

$$\begin{pmatrix} ct' & \\ & \mathbf{r}' \end{pmatrix} = \begin{pmatrix} \gamma_+ & \\ & -\boldsymbol{\gamma}_0 \end{pmatrix}\begin{pmatrix} ct & \\ & \mathbf{r} \end{pmatrix}\begin{pmatrix} \gamma_+ & \\ & -\boldsymbol{\gamma}_0 \end{pmatrix}, \quad \boldsymbol{\gamma}_0 = \gamma_- \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} .$$

More generally, when a particle moves at a speed $\mathbf{v}$ with direction cosine (A, B, C), then we have the **Lorentz form** [5)] as follows:

(i) The transformation of **coordinate matrix-vector**[1,5] and its abbreviation are

$$\begin{pmatrix} ct' & \\ & -\mathbf{r}' \end{pmatrix} = \begin{pmatrix} \gamma_+ & \\ & \underline{\boldsymbol{\gamma}_0} \end{pmatrix} \begin{pmatrix} ct & \\ & -\mathbf{r} \end{pmatrix} \begin{pmatrix} \gamma_+ & \\ & \underline{\boldsymbol{\gamma}_0} \end{pmatrix} = {}^{+}\begin{pmatrix} ct & \\ & -\mathbf{r} \end{pmatrix}^{+}, \quad \boldsymbol{\gamma}_0 = \gamma_- \begin{pmatrix} A \\ B \\ C \end{pmatrix}.$$

(ii) The transformation of **derivative matrix-vector**[1,5] and its abbreviation are

$$\begin{pmatrix} \dfrac{\partial}{\partial ct'} & \\ & -\dfrac{\partial}{\partial \mathbf{r}'} \end{pmatrix} = \begin{pmatrix} \gamma_+ & \\ & -\underline{\boldsymbol{\gamma}_0} \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial ct} & \\ & -\dfrac{\partial}{\partial \mathbf{r}} \end{pmatrix} \begin{pmatrix} \gamma_+ & \\ & -\underline{\boldsymbol{\gamma}_0} \end{pmatrix} = {}^{-}\begin{pmatrix} \dfrac{\partial}{\partial ct} & \\ & -\dfrac{\partial}{\partial \mathbf{r}} \end{pmatrix}^{-}.$$

(iii) The transformation of **potential matrix-vector**[1,5] and its abbreviation are

$$\begin{pmatrix} \phi' & \\ & -c\mathbf{A}' \end{pmatrix} = \begin{pmatrix} \gamma_+ & \\ & \underline{\boldsymbol{\gamma}_0} \end{pmatrix} \begin{pmatrix} \phi & \\ & -c\mathbf{A} \end{pmatrix} \begin{pmatrix} \gamma_+ & \\ & \underline{\boldsymbol{\gamma}_0} \end{pmatrix} = {}^{+}\begin{pmatrix} \phi & \\ & -c\mathbf{A} \end{pmatrix}^{+}.$$

And we call them a **Lorentz form.**

Using this Lorentz form and the relation (＊＊＊), we get

$$\left[{}^{+}\begin{pmatrix} \dfrac{\varepsilon}{c} & \\ & \mathbf{p} \end{pmatrix}^{+} {}^{-}\begin{pmatrix} \delta ct & \\ & \delta\mathbf{r} \end{pmatrix}^{-} \right]_{\mathbf{T}} = \left[{}^{+}\begin{pmatrix} \delta ct & \\ & -\delta\mathbf{r} \end{pmatrix}^{+} {}^{-}\begin{pmatrix} \dfrac{\varepsilon}{c} & \\ & -\mathbf{p} \end{pmatrix}^{-} \right]_{\mathbf{T}},$$

and

$$\left[{}^{+}\begin{pmatrix} \phi & \\ & -c\mathbf{A} \end{pmatrix}^{+} {}^{-}\begin{pmatrix} \delta ct & \\ & \delta\mathbf{r} \end{pmatrix}^{-} \right]_{\mathbf{T}} = \left[{}^{+}\begin{pmatrix} \delta ct & \\ & -\delta\mathbf{r} \end{pmatrix}^{+} {}^{-}\begin{pmatrix} \phi & \\ & c\mathbf{A} \end{pmatrix}^{-} \right]_{\mathbf{T}} \quad \cdots (＊＊＊)'.$$

(II) The variation of the matrix-vector.

Using the relation (＊＊), we get

$$\delta \left[{}^{+}\begin{pmatrix} \dfrac{u_0}{c} & \\ & -\dfrac{\mathbf{u}}{c} \end{pmatrix}^{+} {}^{-}\begin{pmatrix} \dfrac{u_0}{c} & \\ & \dfrac{\mathbf{u}}{c} \end{pmatrix}^{-} \right]_{\mathbf{T}},$$

$$= {}^{+}\begin{pmatrix} \delta(\frac{u_0}{c})\frac{u_0}{c} - \delta(\frac{\mathbf{u}}{c})\frac{\mathbf{u}}{c} & \\ & \delta(\frac{u_0}{c})\frac{\mathbf{u}}{c} - \delta(\frac{\mathbf{u}}{c})\frac{u_0}{c} + i\delta(\frac{\mathbf{u}}{c})\times\frac{\mathbf{u}}{c} \end{pmatrix}^{-}_{\mathrm{T}},$$

$$= \begin{pmatrix} 0 & \\ & \mathbf{0} \end{pmatrix} \cdots (**)' \quad .$$

And the variation of the **potential matrix-vector**[1)] is

$$\delta\, {}^{+}\begin{pmatrix} \phi & \\ & -c\mathbf{A} \end{pmatrix}^{+} = {}^{+}\begin{pmatrix} \delta ct & \\ & -\delta\mathbf{r} \end{pmatrix}^{+} \left[{}^{-}\begin{pmatrix} \frac{\partial}{\partial ct} & \\ & -\frac{\partial}{\partial \mathbf{r}} \end{pmatrix}^{-} {}^{+}\begin{pmatrix} \phi & \\ & -c\mathbf{A} \end{pmatrix}^{+} \right],$$

$$= {}^{+}\begin{pmatrix} \delta ct & \\ & -\delta\mathbf{r} \end{pmatrix}^{+} {}^{-}\begin{pmatrix} E_0 & \\ & \mathbf{E} - ic\mathbf{B} \end{pmatrix}^{+} \cdots (*)' \quad .$$

Another representation of this variation is

$$\delta\, {}^{+}\begin{pmatrix} \phi & \\ & -c\mathbf{A} \end{pmatrix}^{+} = \left[{}^{+}\begin{pmatrix} \delta ct & \\ & -\delta\mathbf{r} \end{pmatrix}^{+} {}^{-}\begin{pmatrix} \frac{\partial}{\partial ct} & \\ & -\frac{\partial}{\partial \mathbf{r}} \end{pmatrix}^{-} \right] {}^{+}\begin{pmatrix} \phi & \\ & -c\mathbf{A} \end{pmatrix}^{+},$$

$$= {}^{+}\begin{pmatrix} \delta ct\frac{\partial}{\partial ct} + \delta\mathbf{r}\cdot\frac{\partial}{\partial \mathbf{r}} & \\ & -\delta ct\frac{\partial}{\partial \mathbf{r}} - \delta\mathbf{r}\frac{\partial}{\partial ct} - i\delta\mathbf{r}\times\frac{\partial}{\partial \mathbf{r}} \end{pmatrix}^{-} {}^{+}\begin{pmatrix} \phi & \\ & -c\mathbf{A} \end{pmatrix}^{+},$$

$$= {}^{+}\begin{pmatrix} \begin{array}{l} \underline{(\delta ct\frac{\partial\phi}{\partial ct} + \delta\mathbf{r}\cdot\mathbf{grad}\phi)} \\ +(\delta ct\, div\, c\mathbf{A} + \delta\mathbf{r}\cdot\frac{\partial c\mathbf{A}}{\partial ct}) \\ +i(\delta\mathbf{r}\times\frac{\partial}{\partial \mathbf{r}})\cdot\mathbf{A} \end{array} & \\ & \begin{array}{l} \underline{-(\delta ct\frac{\partial c\mathbf{A}}{\partial ct} + (\delta\mathbf{r}\cdot\frac{\partial}{\partial \mathbf{r}})c\mathbf{A})} \\ -\{\delta ct\,\mathbf{grad}\phi + \delta\mathbf{r}\frac{\partial\phi}{\partial ct} + i\delta\mathbf{r}\times\mathbf{grad}\phi\} \\ -i\{\delta ct\,\mathbf{rot}\,c\mathbf{A} + \delta\mathbf{r}\times\frac{\partial c\mathbf{A}}{\partial ct} + i(\delta\mathbf{r}\times\frac{\partial}{\partial \mathbf{r}})\times c\mathbf{A}\} \end{array} \end{pmatrix}^{+} .$$

where the **underlined parts** are the usual variation (*).

This variation is the extension of the usual variation.

In this variation, some new features appear as follows:

(Ⅰ) The variation of energy.

(i) The terms $\delta ct(\frac{\partial\phi}{\partial ct}+divc\mathbf{A})=E_0\delta ct$ and $\delta\mathbf{r}\cdot(\mathbf{grad}\phi+\frac{\partial c\mathbf{A}}{\partial ct})=-\mathbf{E}\cdot\delta\mathbf{r}$ are the variation of energy of the charge.

(ii) The term $i(\delta\mathbf{r}\times\frac{\partial}{\partial\mathbf{r}})\cdot c\mathbf{A}=i(\frac{\partial}{\partial\mathbf{r}}\times c\mathbf{A})\cdot\delta\mathbf{r}=i\mathbf{B}\cdot\delta\mathbf{r}$ is the variation of energy of the magnetic charge, because we put $\phi_m=\int\mathbf{B}\cdot\delta\mathbf{r}$, then $i\mathbf{B}=\mathbf{grad}i\phi_m(=i\mathbf{rot}c\mathbf{A})$ is a force of the magnetic charge

(Ⅱ) The variation of momentum.

(i) The term $\delta ct(\mathbf{grad}\phi+\frac{\partial c\mathbf{A}}{\partial ct})=\mathbf{E}\delta ct$ and the term

$$\delta\mathbf{r}\frac{\partial\phi}{\partial ct}+(\delta\mathbf{r}\cdot\frac{\partial}{\partial\mathbf{r}})\mathbf{A}-(\delta\mathbf{r}\times\frac{\partial}{\partial\mathbf{r}})\times c\mathbf{A},$$

$$=\delta\mathbf{r}(\frac{\partial\phi}{\partial ct}+divc\mathbf{A})-\delta\mathbf{r}\times(\mathbf{rot}c\mathbf{A}),$$

$$=E_0\delta ct\frac{d\mathbf{r}}{dct}+\mathbf{B}\times\frac{d\mathbf{r}}{dct}\delta ct$$

are both the variation of momentum of the charge.

(ii) The term $-i\{\delta\mathbf{r}\times(\mathbf{grad}\phi+\frac{\partial c\mathbf{A}}{\partial ct})+\delta ct\mathbf{rot}c\mathbf{A}\}=i(\mathbf{E}\times\frac{d\mathbf{r}}{dct}-\mathbf{B})\delta ct$ is the variation of momentum to the magnetic charge, because $i\mathbf{E}\times\frac{d\mathbf{r}}{dct}$ is a force of the moving magnetic charge.

§ 3 The equation of motion(matrix).

We use the variational method, and can get the following theorem.

Theorem (The equation of motion represented by the matrix-vector)

We define the Lagrangian and its action in the **matrix-vector** form as follows:

$$\mathbf{L}_0 = \left\{ mc \, {}^{+}\!\begin{pmatrix} \frac{u_0}{c} & \\ & -\frac{\mathbf{u}}{c} \end{pmatrix}^{+} + \frac{q}{c} \, {}^{+}\!\begin{pmatrix} \phi & \\ & -c\mathbf{A} \end{pmatrix}^{+} \right\}^{-} \, {}^{-}\!\begin{pmatrix} \frac{u_0}{c} & \\ & \frac{\mathbf{u}}{c} \end{pmatrix}^{-},$$

$$\mathbf{I} = \int_{s_0}^{s_1} \left(\left\{ mc \, {}^{+}\!\begin{pmatrix} \frac{u_0}{c} & \\ & -\frac{\mathbf{u}}{c} \end{pmatrix}^{+} + \frac{q}{c} \, {}^{+}\!\begin{pmatrix} \phi & \\ & -c\mathbf{A} \end{pmatrix}^{+} \right\}^{-} \, {}^{-}\!\begin{pmatrix} \frac{u_0}{c} & \\ & \frac{\mathbf{u}}{c} \end{pmatrix}^{-} \right)_{\mathbf{T}} ds \, .$$

Then we get the equation of motion represented by the matrix-vector

$$\frac{d}{ds} \, {}^{-}\!\left[\begin{pmatrix} \frac{\varepsilon}{c} & \\ & \mathbf{p} \end{pmatrix} + \frac{q}{c} \begin{pmatrix} \phi & \\ & c\mathbf{A} \end{pmatrix} \right]^{-} = \frac{q}{c} \, {}^{-}\!\begin{pmatrix} E_0 & \\ & \mathbf{E} - ic\mathbf{B} \end{pmatrix}^{+} \, {}^{-}\!\begin{pmatrix} \frac{u_0}{c} & \\ & \frac{\mathbf{u}}{c} \end{pmatrix}^{-},$$

where q is the charge and m is the mass.

We define

$$\mathbf{P} = {}^{+}\!\begin{pmatrix} \frac{\varepsilon}{c} & \\ & -\mathbf{p} \end{pmatrix}^{+} = m \, {}^{+}\!\begin{pmatrix} u_0 & \\ & -\mathbf{u} \end{pmatrix}^{+}, \quad \tilde{\mathbf{A}} = {}^{+}\!\begin{pmatrix} \phi & \\ & -c\mathbf{A} \end{pmatrix}^{+} \text{ and } \mathbf{U} = {}^{-}\!\begin{pmatrix} \frac{u_0}{c} & \\ & \frac{\mathbf{u}}{c} \end{pmatrix}^{-}.$$

Then we can put the Lagrangian and its action in the **matrix-vector** form as follows:

$$\mathbf{L}_0 = \left\{ mc \, {}^{+}\!\begin{pmatrix} \frac{u_0}{c} & \\ & -\frac{\mathbf{u}}{c} \end{pmatrix}^{+} + \frac{q}{c} \, {}^{+}\!\begin{pmatrix} \phi & \\ & -c\mathbf{A} \end{pmatrix}^{+} \right\}^{-} \, {}^{-}\!\begin{pmatrix} \frac{u_0}{c} & \\ & \frac{\mathbf{u}}{c} \end{pmatrix}^{-},$$

$$\mathbf{I} = \int_{s_0}^{s_1} \left(\left\{ mc \, {}^{+}\!\begin{pmatrix} \frac{u_0}{c} & \\ & -\frac{\mathbf{u}}{c} \end{pmatrix}^{+} + \frac{q}{c} \, {}^{+}\!\begin{pmatrix} \phi & \\ & -c\mathbf{A} \end{pmatrix}^{+} \right\}^{-} \, {}^{-}\!\begin{pmatrix} \frac{u_0}{c} & \\ & \frac{\mathbf{u}}{c} \end{pmatrix}^{-} \right)_{\mathbf{T}} ds \, .$$

This can be justified as

(i) $\varepsilon(Energy)=\mathbf{P}\cdot\mathbf{v}-L \Leftrightarrow \begin{pmatrix} L & \\ & \mathbf{0} \end{pmatrix} = -\left\{ {}^{+}\begin{pmatrix} \frac{\varepsilon}{c} & \\ & -\mathbf{p} \end{pmatrix}^{+} \, {}^{-}\begin{pmatrix} c & \\ & \mathbf{v} \end{pmatrix}^{-} \right\}_{\mathbf{T}}$, when charge $q=0$.

(Lagrangian)=(Energy-Momentum)×(Velocity)

(ii) $L_0(\text{Lagrangian})=-mc+\frac{q}{c}(c\mathbf{A}\cdot\frac{\mathbf{u}}{c}-\phi\frac{u_0}{c})$

$$\Leftrightarrow \begin{pmatrix} L_0 & \\ & \mathbf{0} \end{pmatrix} = -\left[\left\{ mc \, {}^{+}\begin{pmatrix} \frac{u_0}{c} & \\ & -\frac{\mathbf{u}}{c} \end{pmatrix}^{+} + \frac{q}{c} \, {}^{+}\begin{pmatrix} \phi & \\ & -c\mathbf{A} \end{pmatrix}^{+} \right\} {}^{-}\begin{pmatrix} \frac{u_0}{c} & \\ & \frac{\mathbf{u}}{c} \end{pmatrix}^{-} \right]_{\mathbf{T}} .$$

(Lagrangian)= (Energy-Momentum)×(Velocity)

Then the variation of the action $\mathbf{I}$ is

$$\delta\mathbf{I} = -\delta\int_{s_0}^{s_1} \left({}^{+}\left[mc\begin{pmatrix} \frac{u_0}{c} & \\ & -\frac{\mathbf{u}}{c} \end{pmatrix} + \frac{q}{c}\begin{pmatrix} \phi & \\ & -c\mathbf{A} \end{pmatrix} \right]^{+} \, {}^{-}\begin{pmatrix} \frac{u_0}{c} & \\ & \frac{\mathbf{u}}{c} \end{pmatrix}^{-} \right)_{\mathbf{T}} ds ,$$

$$= -\int_{s_0}^{s_1} \left(mc\delta \, {}^{+}\begin{pmatrix} \frac{u_0}{c} & \\ & -\frac{\mathbf{u}}{c} \end{pmatrix}^{+} \, {}^{-}\begin{pmatrix} \frac{u_0}{c} & \\ & \frac{\mathbf{u}}{c} \end{pmatrix}^{-} ds \right)_{\mathbf{T}}$$

$$-\int_{s_0}^{s_1} \left(\frac{q}{c}\delta \, {}^{+}\begin{pmatrix} \phi & \\ & -c\mathbf{A} \end{pmatrix}^{+} \, {}^{-}\begin{pmatrix} \frac{u_0}{c} & \\ & \frac{\mathbf{u}}{c} \end{pmatrix}^{-} ds \right)_{\mathbf{T}}$$

$$-\int_{s_0}^{s_1} \left[\left\{ {}^{+}\left[mc\begin{pmatrix} \frac{u_0}{c} & \\ & -\frac{\mathbf{u}}{c} \end{pmatrix} + \frac{q}{c}\begin{pmatrix} \phi & \\ & -c\mathbf{A} \end{pmatrix} \right]^{+} \right\} \delta \left\{ {}^{-}\begin{pmatrix} \frac{u_0}{c} & \\ & \frac{\mathbf{u}}{c} \end{pmatrix}^{-} ds \right\} \right]_{\mathbf{T}} .$$

Therefore, using the relations (＊)', (＊＊)' and integration by parts, we get

$$\delta\mathbf{I} = -\int_{s_0}^{s_1}\left[\frac{q}{c}\left\{{}^{+}\begin{pmatrix}\delta ct & \\ & -\delta\mathbf{r}\end{pmatrix}^{+}\,{}^{-}\begin{pmatrix}E_0 & \\ & \mathbf{E}-ic\mathbf{B}\end{pmatrix}^{+}\right\}\,{}^{-}\begin{pmatrix}\frac{u_0}{c} & \\ & \frac{\mathbf{u}}{c}\end{pmatrix}^{-}\right]_{\mathbf{T}} ds$$

$$+\int_{s_0}^{s_1}\left[\frac{d}{ds}\left\{{}^{+}\begin{pmatrix}\frac{\varepsilon}{c} & \\ & -\mathbf{p}\end{pmatrix}^{+} + \frac{q}{c}\,{}^{+}\begin{pmatrix}\phi & \\ & -c\mathbf{A}\end{pmatrix}^{+}\right\}\,{}^{-}\begin{pmatrix}\delta ct & \\ & \delta\mathbf{r}\end{pmatrix}^{-}\right]_{\mathbf{T}} ds$$

$$-\left[\left[\left\{{}^{+}\begin{pmatrix}\frac{\varepsilon}{c} & \\ & -\mathbf{p}\end{pmatrix}^{+} + \frac{q}{c}\,{}^{+}\begin{pmatrix}\phi & \\ & -c\mathbf{A}\end{pmatrix}^{+}\right\}\,{}^{-}\begin{pmatrix}\delta ct & \\ & \delta\mathbf{r}\end{pmatrix}^{-}\right]_{\mathbf{T}}\right]_{s_0}^{s_1}.$$

Using the relation (＊＊＊)’ and the condition of variation as

$$\begin{pmatrix}\delta ct & \\ & \delta\mathbf{r}\end{pmatrix}(s_0) = \begin{pmatrix}\delta ct & \\ & \delta\mathbf{r}\end{pmatrix}(s_1) = \begin{pmatrix}0 & \\ & \mathbf{0}\end{pmatrix}.$$

Lastly we get the following formula:

$$\delta\mathbf{I} = \int_{s_0}^{s_1}\left[{}^{+}\begin{pmatrix}\delta ct & \\ & -\delta\mathbf{r}\end{pmatrix}^{+}\left(\frac{d}{ds}\left\{{}^{-}\begin{pmatrix}\frac{\varepsilon}{c} & \\ & \mathbf{p}\end{pmatrix}^{-} + \frac{q}{c}\,{}^{-}\begin{pmatrix}\phi & \\ & c\mathbf{A}\end{pmatrix}^{-}\right\} - \frac{q}{c}\,{}^{-}\begin{pmatrix}E_0 & \\ & \mathbf{E}-ic\mathbf{B}\end{pmatrix}^{+}\,{}^{-}\begin{pmatrix}\frac{u_0}{c} & \\ & \frac{\mathbf{u}}{c}\end{pmatrix}^{-}\right)\right]_{\mathbf{T}} ds.$$

And this variation $\delta\mathbf{I}$ is always zero to any variation $\begin{pmatrix}\delta ct & \\ & -\delta\mathbf{r}\end{pmatrix}$.

(i) Especially when $\delta\mathbf{r}=\mathbf{0}$, the variation $\delta\mathbf{I}$ is always zero to any variation δct. Therefore

$$\left[\frac{d}{ds}\left\{{}^{-}\begin{pmatrix}\frac{\varepsilon}{c} & \\ & \mathbf{p}\end{pmatrix}^{-} + \frac{q}{c}\,{}^{-}\begin{pmatrix}\phi & \\ & c\mathbf{A}\end{pmatrix}^{-}\right\} - \frac{q}{c}\,{}^{-}\begin{pmatrix}E_0 & \\ & \mathbf{E}-ic\mathbf{B}\end{pmatrix}^{+}\,{}^{-}\begin{pmatrix}\frac{u_0}{c} & \\ & \frac{\mathbf{u}}{c}\end{pmatrix}^{-}\right]_{\mathbf{T}} = \begin{pmatrix}0 & \\ & \mathbf{0}\end{pmatrix} \cdot\cdot \text{(A)}.$$

Especially when $\delta ct = 0$, the variation $\delta\mathbf{I}$ is always zero to any variation $\delta\mathbf{r}$. Therefore

$$\left[\frac{d}{ds}\left\{{}^{-}\begin{pmatrix}\frac{\varepsilon}{c} & \\ & \mathbf{p}\end{pmatrix}^{-}+\frac{q}{c}\,{}^{-}\begin{pmatrix}\phi & \\ & c\mathbf{A}\end{pmatrix}^{-}\right\}-\frac{q}{c}\,{}^{-}\begin{pmatrix}E_0 & \\ & \mathbf{E}-ic\mathbf{B}\end{pmatrix}^{+}\,{}^{-}\begin{pmatrix}\frac{u_0}{c} & \\ & \frac{\mathbf{u}}{c}\end{pmatrix}^{-}\right]_{S}=\begin{pmatrix}0 & \\ & \mathbf{0}\end{pmatrix}\cdot\cdot\ \text{(B)}.$$

By the formulas (A) and (B), we get

$$\frac{d}{ds}\left\{{}^{-}\begin{pmatrix}\frac{\varepsilon}{c} & \\ & \mathbf{p}\end{pmatrix}^{-}+\frac{q}{c}\,{}^{-}\begin{pmatrix}\phi & \\ & c\mathbf{A}\end{pmatrix}^{-}\right\}-\frac{q}{c}\,{}^{-}\begin{pmatrix}E_0 & \\ & \mathbf{E}-ic\mathbf{B}\end{pmatrix}^{+}\,{}^{-}\begin{pmatrix}\frac{u_0}{c} & \\ & \frac{\mathbf{u}}{c}\end{pmatrix}^{-}=\begin{pmatrix}0 & \\ & \mathbf{0}\end{pmatrix}.$$

Therefore we get the following equation of motion[7]:

$$\frac{d}{ds}\left\{{}^{-}\begin{pmatrix}\frac{\varepsilon}{c} & \\ & \mathbf{p}\end{pmatrix}^{-}+\frac{q}{c}\,{}^{-}\begin{pmatrix}\phi & \\ & c\mathbf{A}\end{pmatrix}^{-}\right\}=\frac{q}{c}\,{}^{-}\begin{pmatrix}E_0 & \\ & \mathbf{E}-ic\mathbf{B}\end{pmatrix}^{+}\,{}^{-}\begin{pmatrix}\frac{u_0}{c} & \\ & \frac{\mathbf{u}}{c}\end{pmatrix}^{-}.$$

This equation means that the 4-dimensional force is the 4-dimensional **vector product** between the electromagnetic field and the 4-dimensional velocity.

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