A New Form of Equation of Motion for a Moving Charge and the Lagrangian*

Yoshio TAKEMOTO**

Department of Electrical Engineering and Electronics, School of Engineering,
Nippon Bunri University

Abstract

In our previous paper, we presented a new notion, "matrix−vector", which is a vector where the function of matrix product has been added [(8) Y. Takemoto, Bull. of NBU Vol. 34, No. 1 (2006−Mar.) p. 32].

In this paper, as an application of the matrix−vectors, we deduce an equation of motion represented by matrix for a moving charge in an electromagnetic field.

Contents:

In § 1, using a traditional variational method, we deduce (A) the usual 4-dimensional momentum and (B) equation of motion from the Lagrangian. Now we rewrite its momentum and equation into the matrix−vector form.

In § 2, for preliminaries, we review (A) the matrix−vector and (B) its Lorentz form. Now we define the variation of the matrix−vector and investigate its meaning by comparing this variation with the usual one ($\delta\phi$, δA).

In § 3, we denote the Lagrangian by the matrix−vector form and use the variational method. Then we can get the equation of motion which is represented by matrix−vector form.

New features of this equation are

(1) New effects of the time components E_0 of the electric field appear.
(2) The 4−dimensional complex force appears.
(3) The relativistic invariance of the equation is apparent.

§ 1. Introduction

We put the Lagrangians L and L_0 which are for time dt and for proper time $ds=\sqrt{(dct)^2-(d\mathrm{r})^2}$ respectively, that is,

$$L=-mc^2\sqrt{1-\frac{\mathbf{v}^2}{c^2}}+q(\mathbf{A}\cdot\mathbf{v}-\phi),$$

$$L_0=-mc+\frac{q}{c}\left(c\mathbf{A}\cdot\frac{\mathbf{u}}{c}-\phi\frac{u_0}{c}\right),$$

*平成18年11月22日受理

**日本文理大学工学部電気・電子工学科　教授

where $\frac{u_0}{c}=\frac{dct}{ds}=\frac{1}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}=\gamma$, $\frac{\mathbf{u}}{c}=\frac{d\mathbf{r}}{ds}=\frac{\frac{\mathbf{v}}{c}}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}=\gamma\beta$.

The actions of these Lagrangians are as follows:

$$S=\int_a^b\left\{-mc+\frac{q}{c}\left(c\mathbf{A}\cdot\frac{\mathbf{u}}{c}-\phi\frac{u_0}{c}\right)\right\}ds=\int_{t1}^{t2}\left\{-mc^2\sqrt{1-\frac{v^2}{c^2}}+q\,(\mathbf{A}\cdot\mathbf{v}-\phi)\right\}dt.$$

Further we put the variations.

$$\delta\phi=\frac{d\phi}{dct}\delta ct+\mathbf{grad}\phi\cdot\delta\mathbf{r},\quad \delta\mathbf{A}=\frac{d\mathbf{A}}{dct}\delta ct+div\mathbf{A}\delta\mathbf{r}\cdots\cdots\ (*).$$

And we use the relation $u_0{}^2-\mathbf{u}^2=c^2$, then

$$\delta ds=u_0d(\delta ct)-\mathbf{u}d(\delta\mathbf{r}),\quad \delta(u_0)u_0-\delta(\mathbf{u})\mathbf{u}=0\cdots\cdots\ (**).$$

We get (A) the usual generalized momentum and (B) equation of motion to the moving charge q with the mass m in the electromagnetic field as follows:

(A) The generalized momentum is

$$\mathbf{P}=\frac{\partial L}{\partial\mathbf{v}}=\frac{m\mathbf{v}}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}+q\frac{\mathbf{A}}{c}=\mathbf{p}+q\frac{\mathbf{A}}{c}.$$

The generalized energy is

$$E=\mathbf{P}\cdot\mathbf{v}-L=\frac{mc^2}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}+q\phi=\varepsilon+q\phi$$

And the relation between them is

$$(\mathrm{E}-q\phi)^2-(\mathbf{P}c-q\mathbf{A})^2=\varepsilon^2-(\mathbf{p}c)^2=(mc^2)^2$$

(B) The equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\mathbf{v}}\right)=\frac{\partial L}{\partial\mathbf{r}}$$

The left side term is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\mathbf{v}}\right)=\frac{\partial\mathbf{P}}{\partial t}=\frac{d\mathbf{p}}{dt}+\frac{q}{c}\left(\frac{\partial\mathbf{A}}{\partial t}+(\mathbf{v}\cdot\mathbf{grad})\mathbf{A}\right).$$

The right side term is

$$\frac{\partial L}{\partial\mathbf{r}}=\mathbf{grad}L=q\,\mathbf{grad}(\mathrm{A}\cdot\mathrm{v})-q\,\mathbf{grad}\phi=q\left((\mathrm{v}\cdot\mathbf{grad})\mathrm{A}+\mathrm{v}\times\mathrm{rotA}\right)-q\,\mathbf{grad}\phi.$$

Therefore we get the equation of motion of moving charge in the electromagnetic field.

$$\frac{d\mathbf{p}}{dt}=-q\left(\frac{\partial\mathbf{A}}{\partial t}+\mathbf{grad}\phi\right)+q\,\mathbf{v}\times\mathbf{rotA},$$

$$=q\mathbf{E}+q\frac{\mathbf{v}}{c}\times c\mathbf{B}-iq\left[\underline{c\mathbf{B}-\frac{\mathbf{v}}{c}\times\mathbf{E}}\right]$$

where $\mathbf{E}=-\frac{\partial A}{\partial t}-\mathbf{grad}\phi$, $\mathbf{B}=\mathbf{rotA}$.

The underlined imaginary part is the term which we have added.

Using $\varepsilon^2-(\mathbf{p}c)^2=(mc^2)^2$ and $p=\frac{\varepsilon\mathbf{v}}{c^2}$, we get

$$\frac{d\varepsilon}{dt}=\mathbf{v}\cdot\frac{d\mathbf{p}}{dt},$$

$$=\mathbf{v}\cdot\left(q\mathbf{E}+q\frac{\mathbf{v}}{c}\times c\mathbf{B}\right)\ -\mathbf{v}\cdot\underline{iq[c\mathbf{B}-\frac{\mathbf{v}}{c}\times\mathbf{E}]}$$

$$=q\mathbf{v}\cdot\mathbf{E}-i\underline{q\mathbf{v}\cdot c\mathbf{B}}.$$

We can rewrite these equations by using the matrix-vector as follows:

$$\frac{d}{dt}\begin{pmatrix}\frac{\varepsilon}{c} & \\ & \mathbf{p}\end{pmatrix}=\begin{pmatrix}\frac{q}{c}\mathbf{v}\cdot\mathbf{E}-i\underline{\frac{q}{c}\mathbf{v}\cdot c\mathbf{B}} & \\ & q\mathbf{E}+q\frac{\mathbf{v}}{c}\times c\mathbf{B}-iq\left[\underline{c\mathbf{B}-\frac{\mathbf{v}}{c}\times\mathbf{E}}\right]\end{pmatrix}$$

$$=\frac{q}{c}\begin{pmatrix}0 & \\ & \mathbf{E}-ic\mathbf{B}\end{pmatrix}\begin{pmatrix}c & \\ & \mathbf{v}\end{pmatrix}$$

§ 2. Preliminaries and notations.

In this section, we review (I)-(A) a matrix-vector and (B) its Lorentz form. Now we define (II) the variation of the matrix-vector.

(I)-(A) A matrix-vector. [6)8)]

We identify the 4-dimensional vector $\begin{pmatrix}A_t\\ \mathbf{A}\end{pmatrix}=\begin{pmatrix}A_t\\A_x\\A_y\\A_z\end{pmatrix}\in\mathbf{R}^4$ and the $u(1)$- matrix[2)3)] $\begin{pmatrix}A_t+A_x & A_y+iA_z\\ A_y-iA_z & A_t-A_x\end{pmatrix}$, and we represent this matrix by a symbol $\begin{pmatrix}A_t & \\ & \mathbf{A}\end{pmatrix}=\begin{pmatrix}A_t & \\ & (A_x\ A_y\ A_z)\end{pmatrix}$ and call it a matrix-vector.

And we complexify the each component A_t, A_x, A_y, A_z, that is, we define the symbol

$\begin{pmatrix}A_t & \\ & \mathbf{A}\end{pmatrix}=\begin{pmatrix}A_t & \\ & (A_x\ A_y\ A_z)\end{pmatrix}$ as the matrix $\begin{pmatrix}A_t+A_x & A_y+iA_z\\ A_y-iA_z & A_t-A_x\end{pmatrix}$ with complex components.

Then the product (4-dimensional vector product) between two matrix-vector is as follows:

$$\begin{pmatrix}A_t & \\ & \mathbf{A}\end{pmatrix}\begin{pmatrix}B_t & \\ & \mathbf{B}\end{pmatrix}=\begin{pmatrix}A_tB_t+\underline{\mathbf{A}\cdot\mathbf{B}} & \\ & A_t\mathbf{B}+\mathbf{A}B_t-i\,(\underline{\mathbf{A}\times\mathbf{B}})\end{pmatrix}$$

And we define $\begin{pmatrix}A_t & \\ & \mathbf{A}\end{pmatrix}_T$, $\begin{pmatrix}A_t & \\ & \mathbf{A}\end{pmatrix}_S$ and $\overline{\begin{pmatrix}A_t & \\ & \mathbf{A}\end{pmatrix}}$ are each the time part, space part and a conjugate[4)] of $\begin{pmatrix}A_t & \\ & \mathbf{A}\end{pmatrix}$ respectively.

This conjugate corresponds to the cofactor matrix of matrix $\begin{pmatrix}A_t+A_x & A_y+iA_z\\ A_y-iA_z & A_t-A_x\end{pmatrix}$.

Therefore we get the relation:

$$\overline{\begin{pmatrix}A_t & \\ & \mathbf{A}\end{pmatrix}\begin{pmatrix}B_t & \\ & \mathbf{B}\end{pmatrix}}=\overline{\begin{pmatrix}B_t & \\ & \mathbf{B}\end{pmatrix}}\ \overline{\begin{pmatrix}A_t & \\ & \mathbf{A}\end{pmatrix}},\ \text{and}\ \left[\begin{pmatrix}A_t & \\ & \mathbf{A}\end{pmatrix}\begin{pmatrix}B_t & \\ & \mathbf{B}\end{pmatrix}\right]_T=\left[\overline{\begin{pmatrix}B_t & \\ & \mathbf{B}\end{pmatrix}}\ \overline{\begin{pmatrix}A_t & \\ & \mathbf{A}\end{pmatrix}}\right]_T\cdots\cdots(***).$$

(B) The Lorentz form.[8]

When a particle moves to the x-direction at the speed v, then we have the Lorentz transformation:

$$\begin{cases} ct' = \gamma(ct-\beta x) \\ x' = \gamma(x-\beta ct) \\ y' = y \\ z' = z \end{cases}$$

where $\gamma = \dfrac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \cosh\Theta$ and $\gamma\beta = \dfrac{\frac{v}{c}}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \sinh\Theta$.

And we can rewrite this transformation by using the matrix :

$$\begin{pmatrix} ct'+x' & y'+iz' \\ y'-iz' & ct'-x' \end{pmatrix} = \begin{pmatrix} \gamma(1-\beta)(ct+x) & y+iz \\ y-iz & \gamma(1+\beta)(ct-x) \end{pmatrix},$$

$$= \begin{pmatrix} \gamma_+ - \gamma_- & 0 \\ 0 & \gamma_+ + \gamma_- \end{pmatrix} \begin{pmatrix} ct+x & y+iz \\ y-iz & ct-x \end{pmatrix} \begin{pmatrix} \gamma_+ - \gamma_- & 0 \\ 0 & \gamma_+ + \gamma_- \end{pmatrix},$$

where $\gamma_+ = \sqrt{\dfrac{\gamma+1}{2}} = \cosh\dfrac{\Theta}{2}$ and $\gamma_- = \sqrt{\dfrac{\gamma-1}{2}} = \sinh\dfrac{\Theta}{2}$.

Then we have a relativistic transformation in the matrix-vector form:

$$\begin{pmatrix} ct' & \\ & \mathbf{r}' \end{pmatrix} = \begin{pmatrix} \gamma_+ & \\ & -\gamma_0 \end{pmatrix} \begin{pmatrix} ct & \\ & \mathbf{r} \end{pmatrix} \begin{pmatrix} \gamma_+ & \\ & -\gamma_0 \end{pmatrix},\ \gamma_0 = \gamma_- \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

More generally, when a particle moves at a speed $\mathbf{v}$ with direction cosine (A, B, C), then we have the Lorentz form[5] as follows:

(i) The transformation of coordinate matrix-vector[1,5] and its abbreviation are

$$\begin{pmatrix} ct' & \\ & -\mathbf{r}' \end{pmatrix} = \begin{pmatrix} \gamma_+ & \\ & \underline{\gamma_0} \end{pmatrix} \begin{pmatrix} ct & \\ & -\mathbf{r} \end{pmatrix} \begin{pmatrix} \gamma_+ & \\ & \underline{\gamma_0} \end{pmatrix} = {}^{+}\begin{pmatrix} ct & \\ & -\mathbf{r} \end{pmatrix}^{+},\ \gamma_0 = \gamma_- \begin{pmatrix} A \\ B \\ C \end{pmatrix}.$$

(ii) The transformation of derivative matrix-vector[1,5] and its abbreviation are

$$\begin{pmatrix} \frac{\partial}{\partial ct'} & \\ & -\frac{\partial}{\partial \mathbf{r}'} \end{pmatrix} = \begin{pmatrix} \gamma^+ & \\ & -\gamma_0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial ct} & \\ & -\frac{\partial}{\partial \mathbf{r}} \end{pmatrix} \begin{pmatrix} \gamma^+ & \\ & -\underline{\gamma_0} \end{pmatrix} = {}^{-}\begin{pmatrix} \frac{\partial}{\partial ct} & \\ & -\frac{\partial}{\partial \mathbf{r}} \end{pmatrix}^{-}.$$

(iii) The transformation of potential matrix-vector[1,5] and its abbreviation are

$$\begin{pmatrix} \phi' & \\ & -c\mathbf{A}' \end{pmatrix} = \begin{pmatrix} \gamma_+ & \\ & \underline{\gamma_0} \end{pmatrix} \begin{pmatrix} \phi & \\ & -c\mathbf{A} \end{pmatrix} \begin{pmatrix} \gamma_+ & \\ & \underline{\gamma_0} \end{pmatrix} = {}^{+}\begin{pmatrix} \phi & \\ & -c\mathbf{A} \end{pmatrix}^{+}.$$

And we call them a **Lorentz form.**

Using this Lorentz form and the relation (＊＊＊), we get

$$\left[{}^{+}\begin{pmatrix} \frac{\varepsilon}{c} & \\ & \mathbf{p} \end{pmatrix}^{+-}\begin{pmatrix} \delta ct & \\ & \delta\mathbf{r} \end{pmatrix}^{-}\right]_T = \left[{}^{+}\begin{pmatrix} \delta ct & \\ & -\delta\mathbf{r} \end{pmatrix}^{+-}\begin{pmatrix} \frac{\varepsilon}{c} & \\ & -\mathbf{p} \end{pmatrix}^{-}\right]_T,$$

and

$$\left[{}^{+}\begin{pmatrix} \phi & \\ & -c\mathbf{A} \end{pmatrix}^{+-}\begin{pmatrix} \delta ct & \\ & \delta\mathbf{r} \end{pmatrix}^{-}\right]_T = \left[{}^{+}\begin{pmatrix} \delta ct & \\ & -\delta\mathbf{r} \end{pmatrix}^{+-}\begin{pmatrix} \phi & \\ & c\mathbf{A} \end{pmatrix}^{-}\right]_T \cdots\cdots (***)'.$$

(II) The variation of the matrix-vector.

Using the relation (＊＊), we get

$$\delta\left[{}^{+}\begin{pmatrix} \frac{u_0}{c} & \\ & -\frac{\mathbf{u}}{c} \end{pmatrix}^{+-}\begin{pmatrix} \frac{u_0}{c} & \\ & \frac{\mathbf{u}}{c} \end{pmatrix}^{-}\right]_T,$$

$$= {}^{+}\begin{pmatrix} \delta\left(\frac{u_0}{c}\right)\frac{u_0}{c} - \delta\left(\frac{\mathbf{u}}{c}\right)\frac{\mathbf{u}}{c} & \\ & \delta\left(\frac{u_0}{c}\right)\frac{\mathbf{u}}{c} - \delta\left(\frac{\mathbf{u}}{c}\right)\frac{u_0}{c} + i\delta\left(\frac{\mathbf{u}}{c}\right)\times\frac{\mathbf{u}}{c} \end{pmatrix}^{-}_T,$$

$$= \begin{pmatrix} 0 & \\ & \mathbf{0} \end{pmatrix} \cdots\cdots (**)'.$$

And the variation of the potential matrix-vector[1] is

$$\delta\,{}^{+}\begin{pmatrix} \phi & \\ & -c\mathbf{A} \end{pmatrix}^{+} = {}^{+}\begin{pmatrix} \delta ct & \\ & -\delta\mathbf{r} \end{pmatrix}^{+}\left[{}^{-}\begin{pmatrix} \frac{\partial}{\partial ct} & \\ & -\frac{\partial}{\partial \mathbf{r}} \end{pmatrix}^{-+}\begin{pmatrix} \phi & \\ & -c\mathbf{A} \end{pmatrix}^{+}\right],$$

$$= {}^{+}\begin{pmatrix} \delta ct & \\ & -\delta\mathbf{r} \end{pmatrix}^{+-}\begin{pmatrix} E_0 & \\ & \mathbf{E}-ic\mathbf{B} \end{pmatrix}^{+} \cdots\cdots (*)'.$$

Another representation of this variation is

$$\delta\,{}^{+}\begin{pmatrix} \phi & \\ & -c\mathbf{A} \end{pmatrix}^{+} = \left[{}^{+}\begin{pmatrix} \delta ct & \\ & -\delta\mathbf{r} \end{pmatrix}^{+-}\begin{pmatrix} \frac{\partial}{\partial ct} & \\ & -\frac{\partial}{\partial \mathbf{r}} \end{pmatrix}^{-}\right]{}^{+}\begin{pmatrix} \phi & \\ & -c\mathbf{A} \end{pmatrix}^{+}$$

$$= {}^{+}\begin{pmatrix} \delta ct\frac{\partial}{\partial ct} + \delta\mathbf{r}\cdot\frac{\partial}{\partial \mathbf{r}} & \\ & -\delta ct\frac{\partial}{\partial \mathbf{r}} - \delta\mathbf{r}\frac{\partial}{\partial ct} - i\delta\mathbf{r}\times\frac{\partial}{\partial \mathbf{r}} \end{pmatrix}^{-+}\begin{pmatrix} \phi & \\ & -c\mathbf{A} \end{pmatrix}^{+},$$

$$
= {}^{+}\!\begin{pmatrix} \begin{matrix} \underline{\left(\delta ct\dfrac{\partial\phi}{\partial ct}+\delta\mathbf{r}\cdot\mathbf{grad}\phi\right)} \\ +\left(\delta ct div c\mathbf{A}+\delta\mathbf{r}\cdot\dfrac{\partial c\mathbf{A}}{\partial ct}\right) \\ +i\left(\delta\mathbf{r}\times\dfrac{\partial}{\partial\mathbf{r}}\right)\cdot\mathbf{A} \end{matrix} & \\ & \begin{matrix} -\underline{\left(\delta ct\dfrac{\partial c\mathbf{A}}{\partial ct}+\left(\delta\mathbf{r}\cdot\dfrac{\partial}{\partial\mathbf{r}}\right)c\mathbf{A}\right)} \\ -\left\{\delta ct\,\mathbf{grad}\phi+\delta\mathbf{r}\dfrac{\partial\phi}{act}+i\delta\mathbf{r}\times\mathbf{grad}\phi\right\} \\ -i\left\{\delta ct\,\mathbf{rot}c\mathbf{A}+\delta\mathbf{r}\times\dfrac{\partial c\mathbf{A}}{\partial ct}+i\left(\delta\mathbf{r}\times\dfrac{\partial}{\partial\mathbf{r}}\right)\times c\mathbf{A}\right\} \end{matrix} \end{pmatrix}^{+}.
$$

where the underlined parts are the usual variation (∗).

This variation is the extension of the usual variation. In this variation, some new features appear as follows:

(Ⅰ) The variation of energy.

(i) The terms $\delta ct\left(\dfrac{\partial\phi}{\partial ct}+div c\mathbf{A}\right)=E_0\delta ct$ and $\delta\mathbf{r}\cdot\left(\mathbf{grad}\phi+\dfrac{\partial c\mathbf{A}}{\partial ct}\right)=-\mathbf{E}\cdot\delta\mathbf{r}$ are the variation of energy of the charge.

(ii) The term $i\left(\delta\mathbf{r}\times\dfrac{\partial}{\partial\mathbf{r}}\right)\cdot c\mathbf{A}=i\left(\dfrac{\partial}{\partial\mathbf{r}}\times c\mathbf{A}\right)\cdot\delta\mathbf{r}=i\mathbf{B}\cdot\delta\mathbf{r}$ is the variation of energy of the magnetic charge, because we put $\phi_m=\int\mathbf{B}\cdot\delta\mathbf{r}$, then $i\mathbf{B}=\mathbf{grad}i\phi_m$ $(=i\,\mathbf{rot}c\mathbf{A})$ is a force of the magnetic charge

(Ⅱ) The variation of momentum.

(i) The term $\delta ct\left(\mathbf{grad}\phi+\dfrac{\partial c\mathbf{A}}{\partial ct}\right)=\mathbf{E}\delta ct$ and the term

$$
\delta\mathbf{r}\frac{\partial\phi}{\partial ct}+\left(\delta\mathbf{r}\cdot\frac{\partial}{\partial\mathbf{r}}\right)\mathbf{A}-\left(\delta\mathbf{r}\times\frac{\partial}{\partial\mathbf{r}}\right)\times c\mathbf{A},
$$

$$
=\delta\mathbf{r}\left(\frac{\partial\phi}{\partial ct}+div c\mathbf{A}\right)-\delta\mathbf{r}\times(\mathbf{rot}c\mathbf{A}),
$$

$$
=E_0\delta ct\frac{d\mathbf{r}}{dct}+\mathbf{B}\times\frac{d\mathbf{r}}{dct}\delta ct
$$

are both the variation of momentum of the charge.

(ii) The term $-i\left\{\delta\mathbf{r}\times\left(\mathbf{grad}\phi+\dfrac{\partial c\mathbf{A}}{\partial ct}\right)+\delta ct\,\mathbf{rot}c\mathbf{A}\right\}=i\left(\mathbf{E}\times\dfrac{d\mathbf{r}}{dct}-\mathbf{B}\right)\delta ct$ is the variation of momentum to the magnetic charge, because $i\mathbf{E}\times\dfrac{d\mathbf{r}}{dct}$ is a force of the moving magnetic charge.

§ 3. The equation of motion (matrix).

We use the variational method, and can get the following theorem.

Theorem (The equation of motion represented by the matrix-vector)

We define the Lagrangian and its action in the matrix-vector form as follows:

$$
\mathrm{L}_0=\left\{mc\,{}^{+}\!\begin{pmatrix}\frac{u_0}{c} & \\ & -\frac{\mathbf{u}}{c}\end{pmatrix}^{+}+\frac{q}{c}\,{}^{+}\!\begin{pmatrix}\phi & \\ & -c\mathbf{A}\end{pmatrix}^{+}\right\}{}^{-}\!\begin{pmatrix}\frac{u_0}{c} & \\ & \frac{\mathbf{u}}{c}\end{pmatrix}^{-},
$$

$$
\mathrm{I}=\int_{\mathrm{S}_0}^{\mathrm{S}_1}\left(\left\{mc\,{}^{+}\!\begin{pmatrix}\frac{u_0}{c} & \\ & -\frac{\mathbf{u}}{c}\end{pmatrix}^{+}+\frac{q}{c}\,{}^{+}\!\begin{pmatrix}\phi & \\ & -c\mathbf{A}\end{pmatrix}^{+}\right\}{}^{-}\!\begin{pmatrix}\frac{u_0}{c} & \\ & \frac{\mathbf{u}}{c}\end{pmatrix}^{-}\right)_{\mathrm{T}}ds.
$$

Then we get the equation of motion represented by the matrix-vector

$$
\frac{d}{ds}{}^{-}\!\left[\begin{pmatrix}\frac{\varepsilon}{c} & \\ & \mathbf{p}\end{pmatrix}+\frac{q}{c}\begin{pmatrix}\phi & \\ & c\mathbf{A}\end{pmatrix}\right]^{-}=\frac{q}{c}\,{}^{-}\!\begin{pmatrix}E_0 & \\ & \mathbf{E}-ic\mathbf{B}\end{pmatrix}^{+}\,{}^{-}\!\begin{pmatrix}\frac{u_0}{c} & \\ & \frac{\mathbf{u}}{c}\end{pmatrix}^{-},
$$

where q is the charge and m is the mass.

We define

$$
\tilde{\mathrm{P}}={}^{+}\!\begin{pmatrix}\frac{\varepsilon}{c} & \\ & -\mathbf{p}\end{pmatrix}^{+}=m\,{}^{+}\!\begin{pmatrix}u_0 & \\ & -\mathbf{u}\end{pmatrix}^{+},\quad \tilde{\mathrm{A}}={}^{+}\!\begin{pmatrix}\phi & \\ & -c\mathbf{A}\end{pmatrix}^{+}\text{ and }\mathbf{U}={}^{-}\!\begin{pmatrix}\frac{u_0}{c} & \\ & \frac{\mathbf{u}}{c}\end{pmatrix}^{-}.
$$

Then we can put the Lagrangian and its action in the matrix-vector form as follows:

$$
\mathrm{L}_0=\left\{mc\,{}^{+}\!\begin{pmatrix}\frac{u_0}{c} & \\ & -\frac{\mathbf{u}}{c}\end{pmatrix}^{+}+\frac{q}{c}\,{}^{+}\!\begin{pmatrix}\phi & \\ & -c\mathbf{A}\end{pmatrix}^{+}\right\}{}^{-}\!\begin{pmatrix}\frac{u_0}{c} & \\ & \frac{\mathbf{u}}{c}\end{pmatrix}^{-},
$$

$$
\mathrm{I}=\int_{\mathrm{S}_0}^{\mathrm{S}_1}\left(\left\{mc\,{}^{+}\!\begin{pmatrix}\frac{u_0}{c} & \\ & -\frac{\mathbf{u}}{c}\end{pmatrix}^{+}+\frac{q}{c}\,{}^{+}\!\begin{pmatrix}\phi & \\ & -c\mathbf{A}\end{pmatrix}^{+}\right\}{}^{-}\!\begin{pmatrix}\frac{u_0}{c} & \\ & \frac{\mathbf{u}}{c}\end{pmatrix}^{-}\right)_{\mathrm{T}}ds.
$$

This can be justified as

(i) $\varepsilon(Energy)\ =\mathbf{P}\cdot\mathbf{v}-L\Leftrightarrow\begin{pmatrix}L & \\ & \mathbf{0}\end{pmatrix}=-\left\{{}^{+}\!\begin{pmatrix}\frac{\varepsilon}{c} & \\ & -\mathbf{P}\end{pmatrix}^{+}\,{}^{-}\!\begin{pmatrix}c & \\ & \mathbf{v}\end{pmatrix}^{-}\right\}_{\mathrm{T}}$, when charge $q=0$.

(Lagrangian) = (Energy−Momentum) × (Velocity)

(ii) $L_0(\text{Lagrangian})=-mc+\dfrac{q}{c}\left(c\mathbf{A}\cdot\dfrac{\mathbf{u}}{c}-\phi\dfrac{u_0}{c}\right)$

$$
\Leftrightarrow\begin{pmatrix}L_0 & \\ & \mathbf{0}\end{pmatrix}=-\left[\left\{mc\,{}^{+}\!\begin{pmatrix}\frac{u_0}{c} & \\ & -\frac{\mathbf{u}}{c}\end{pmatrix}^{+}+\frac{q}{c}\,{}^{+}\!\begin{pmatrix}\phi & \\ & -c\mathbf{A}\end{pmatrix}^{+}\right\}{}^{-}\!\begin{pmatrix}\frac{u_0}{c} & \\ & \frac{\mathbf{u}}{c}\end{pmatrix}^{-}\right]_{\mathrm{T}}.
$$

(Lagrangian) = (Energy−Momentum) × (Velocity)

Then the variation of the action $\mathbf{I}$ is

$$
\delta\mathbf{I}=-\delta\int_{\mathrm{S}_0}^{\mathrm{S}_1}\left({}^{+}\!\left[mc\begin{pmatrix}\frac{u_0}{c} & \\ & -\frac{\mathbf{u}}{c}\end{pmatrix}+\frac{q}{c}\begin{pmatrix}\phi & \\ & -c\mathbf{A}\end{pmatrix}\right]^{+}\,{}^{-}\!\begin{pmatrix}\frac{u_0}{c} & \\ & \frac{\mathbf{u}}{c}\end{pmatrix}^{-}\right)_{\mathrm{T}}ds,
$$

$$
=-\int_{\mathrm{S}_0}^{\mathrm{S}_1}\left(mc\delta\,{}^{+}\!\begin{pmatrix}\frac{u_0}{c} & \\ & -\frac{\mathbf{u}}{c}\end{pmatrix}^{+}\,{}^{-}\!\begin{pmatrix}\frac{u_0}{c} & \\ & \frac{\mathbf{u}}{c}\end{pmatrix}^{-}ds\right)_{\mathrm{T}}-\int_{\mathrm{S}_0}^{\mathrm{S}_1}\left(\frac{q}{c}\delta\,{}^{+}\!\begin{pmatrix}\phi & \\ & -c\mathbf{A}\end{pmatrix}^{+}\,{}^{-}\!\begin{pmatrix}\frac{u_0}{c} & \\ & \frac{\mathbf{u}}{c}\end{pmatrix}^{-}ds\right)_{\mathrm{T}}
$$

$$
-\int_{\mathrm{S}_0}^{\mathrm{S}_1}\left[\left\{{}^{+}\!\left[mc\begin{pmatrix}\frac{u_0}{c} & \\ & -\frac{\mathbf{u}}{c}\end{pmatrix}+\frac{q}{c}\begin{pmatrix}\phi & \\ & -c\mathbf{A}\end{pmatrix}\right]^{+}\right\}\delta\left\{{}^{-}\!\begin{pmatrix}\frac{u_0}{c} & \\ & \frac{\mathbf{u}}{c}\end{pmatrix}^{-}ds\right\}\right]_{\mathrm{T}}.
$$

Therefore, using the relations (∗)', (∗∗)' and integration by parts, we get

$$\delta\mathbf{I} = -\int_{S_0}^{S_1}\left[\frac{q}{c}\left\{{}^{+}\!\begin{pmatrix}\delta ct & \\ & -\delta\mathbf{r}\end{pmatrix}^{+}\,{}^{-}\!\begin{pmatrix}E_0 & \\ & \mathbf{E}-ic\mathbf{B}\end{pmatrix}^{+}\right\}{}^{-}\!\begin{pmatrix}\frac{u_0}{c} & \\ & \frac{\mathbf{u}}{c}\end{pmatrix}^{-}\right]_{T} ds$$
$$+\int_{S_0}^{S_1}\left[\frac{d}{ds}\left\{{}^{+}\!\begin{pmatrix}\frac{\varepsilon}{c} & \\ & -\mathbf{p}\end{pmatrix}^{+} + \frac{q}{c}\,{}^{+}\!\begin{pmatrix}\phi & \\ & -c\mathbf{A}\end{pmatrix}^{+}\right\}{}^{-}\!\begin{pmatrix}\delta ct & \\ & \delta r\end{pmatrix}^{-}\right]_{T} ds$$
$$-\left[\left[\left\{{}^{+}\!\begin{pmatrix}\frac{\varepsilon}{c} & \\ & -\mathbf{p}\end{pmatrix}^{+} + \frac{q}{c}\,{}^{+}\!\begin{pmatrix}\phi & \\ & -c\mathbf{A}\end{pmatrix}^{+}\right\}{}^{-}\!\begin{pmatrix}\delta ct & \\ & \delta\mathbf{r}\end{pmatrix}^{-}\right]_{T}\right]_{S_0}^{S_1}.$$

Using the relation (＊＊＊)' and the condition of variation as

$$\begin{pmatrix}\delta ct & \\ & \delta\mathbf{r}\end{pmatrix}(S_0) = \begin{pmatrix}\delta ct & \\ & \delta\mathbf{r}\end{pmatrix}(S_1) = \begin{pmatrix}0 & \\ & \mathbf{0}\end{pmatrix}.$$

Lastly we get the following formula:

$$\delta\mathbf{I} = -\int_{S_0}^{S_1}\left[{}^{+}\!\begin{pmatrix}\delta ct & \\ & -\delta\mathbf{r}\end{pmatrix}^{+}\left(\frac{d}{ds}\left\{{}^{-}\!\begin{pmatrix}\frac{\varepsilon}{c} & \\ & \mathbf{p}\end{pmatrix}^{-} + \frac{q}{c}\,{}^{-}\!\begin{pmatrix}\phi & \\ & c\mathbf{A}\end{pmatrix}^{-}\right\} - \frac{q}{c}\,{}^{-}\!\begin{pmatrix}E_0 & \\ & \mathbf{E}-ic\mathbf{B}\end{pmatrix}^{+}\,{}^{-}\!\begin{pmatrix}\frac{u_0}{c} & \\ & \frac{\mathbf{u}}{c}\end{pmatrix}^{-}\right)\right]_{T} ds.$$

And this variation $\delta\mathbf{I}$ is always zero to any variation $\begin{pmatrix}\delta ct & \\ & -\delta\mathbf{r}\end{pmatrix}$.

(i) Especially when $\delta\mathbf{r} = \mathbf{0}$, the variation $\delta\mathbf{I}$ is always zero to any variation δct.
Therefore

$$\left[\frac{d}{ds}\left\{{}^{-}\!\begin{pmatrix}\frac{\varepsilon}{c} & \\ & \mathbf{p}\end{pmatrix}^{-} + \frac{q}{c}\,{}^{-}\!\begin{pmatrix}\phi & \\ & c\mathbf{A}\end{pmatrix}^{-}\right\} - \frac{q}{c}\,{}^{-}\!\begin{pmatrix}E_0 & \\ & \mathbf{E}-ic\mathbf{B}\end{pmatrix}^{+}\,{}^{-}\!\begin{pmatrix}\frac{u_0}{c} & \\ & \frac{\mathbf{u}}{c}\end{pmatrix}^{-}\right]_{T} = \begin{pmatrix}0 & \\ & \mathbf{0}\end{pmatrix} \cdots \text{(A)}.$$

Especially when $\delta ct = 0$, the variation is $\delta\mathbf{I}$ is always zero to any variation $\delta\mathbf{r}$.
Therefore

$$\left[\frac{d}{ds}\left\{{}^{-}\!\begin{pmatrix}\frac{\varepsilon}{c} & \\ & \mathbf{p}\end{pmatrix}^{-} + \frac{q}{c}\,{}^{-}\!\begin{pmatrix}\phi & \\ & c\mathbf{A}\end{pmatrix}^{-}\right\} - \frac{q}{c}\,{}^{-}\!\begin{pmatrix}E_0 & \\ & \mathbf{E}-ic\mathbf{B}\end{pmatrix}^{+}\,{}^{-}\!\begin{pmatrix}\frac{u_0}{c} & \\ & \frac{\mathbf{u}}{c}\end{pmatrix}^{-}\right]_{S} = \begin{pmatrix}0 & \\ & \mathbf{0}\end{pmatrix} \cdots \text{(B)}.$$

By the formulas (A) and (B), we get

$$\frac{d}{ds}\left\{{}^{-}\!\begin{pmatrix}\frac{\varepsilon}{c} & \\ & \mathbf{p}\end{pmatrix}^{-} + \frac{q}{c}\,{}^{-}\!\begin{pmatrix}\phi & \\ & c\mathbf{A}\end{pmatrix}^{-}\right\} - \frac{q}{c}\,{}^{-}\!\begin{pmatrix}E_0 & \\ & \mathbf{E}-ic\mathbf{B}\end{pmatrix}^{+}\,{}^{-}\!\begin{pmatrix}\frac{u_0}{c} & \\ & \frac{\mathbf{u}}{c}\end{pmatrix}^{-} = \begin{pmatrix}0 & \\ & \mathbf{0}\end{pmatrix}.$$

Therefore we get the following equation of motion[7]:

$$\frac{d}{ds}\left\{{}^{-}\!\begin{pmatrix}\frac{\varepsilon}{c} & \\ & \mathbf{p}\end{pmatrix}^{-} + \frac{q}{c}\,{}^{-}\!\begin{pmatrix}\phi & \\ & c\mathbf{A}\end{pmatrix}^{-}\right\} = \frac{q}{c}\,{}^{-}\!\begin{pmatrix}E_0 & \\ & \mathbf{E}-ic\mathbf{B}\end{pmatrix}^{+}\,{}^{-}\!\begin{pmatrix}\frac{u_0}{c} & \\ & \frac{\mathbf{u}}{c}\end{pmatrix}^{-}.$$

This equation means that the 4-dimensional force is the 4-dimensional vector product between the electromagnetic field and the 4-dimensional velocity.

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