Printed by
DAIGAKU LETTERPRESS CO.
Tokaichi-cho, Naka-ku, Hiroshima, Japan

With the Compliments of the Authors

AFFINE STRUCTURES OF MAXIMAL SOLVABLE SUBALGEBRAS OF NON-COMPACT SEMI-SIMPLE LIE ALGEBRAS

By

Yoshio TAKEMOTO and Satoru YAMAGUCHI

Reprinted from the Memoirs of the Faculty of Science,
Kyushu University, Series A, Mathematics,
Volume XXXV No. 1, 1981

FUKUOKA, JAPAN

March, 1981

Memoirs of the Faculty of Science, Kyushu University
Ser. A, Vol. 35, No. 1, 1981

AFFINE STRUCTURES OF MAXIMAL SOLVABLE SUBALGEBRAS OF NON-COMPACT SEMI-SIMPLE LIE ALGEBRAS

By

Yoshio TAKEMOTO* and Satoru YAMAGUCHI

(Received Aug. 23, 1979)
(Revised Jan. 11, 1980)

1. Introduction. This is a continuation of [7]. In what follows, we say for short that a solvable Lie algebra $\mathfrak{s}$ over $\mathbf{R}$ (resp. $\mathbf{C}$) admits a real (resp. complex) *affine structure* if a simply connected real (resp. complex) Lie group with Lie algebra $\mathfrak{s}$ operates simply transitively by real (resp. complex) affine transformations of $\mathbf{R}^n$ (resp. $\mathbf{C}^n$), where $n=\dim_{\mathbf{R}} \mathfrak{s}$ (resp. $\dim_{\mathbf{C}} \mathfrak{s}$).

The purpose of this note is to prove the following

THEOREM. *Let $\tilde{\mathfrak{b}}$ be a Borel subalgebra, i.e., a maximal solvable subalgebra of a complex semi-simple Lie algebra $\tilde{\mathfrak{g}}$ and $\mathfrak{s}$ a maximal solvable subalgebra of a non-compact real semi-simple Lie algebra $\mathfrak{g}$.*

Then we have

1. *$\tilde{\mathfrak{b}}$ admits a complex affine structure.*
2. *$\mathfrak{s}$ admits a real affine structure.*

If there is no danger of confusion, we say simply affine structures without mentioning real or complex structures. Let $\tilde{\mathfrak{l}}$ be a Lie algebra over $\mathbf{C}$. When we regard $\tilde{\mathfrak{l}}$ as a Lie algebra over $\mathbf{R}$, we denote it by $\tilde{\mathfrak{l}}^{\mathbf{R}}$. Then it is easy to see that $\tilde{\mathfrak{b}}^{\mathbf{R}}$ is a maximal solvable subalgebra of $\tilde{\mathfrak{g}}^{\mathbf{R}}$ and conversely for any maximal solvable subalgebra $\mathfrak{s}$ of $\tilde{\mathfrak{g}}^{\mathbf{R}}$ there exists a Borel subalgebra $\tilde{\mathfrak{b}}$ of $\tilde{\mathfrak{g}}$ such that $\mathfrak{s}=\tilde{\mathfrak{b}}^{\mathbf{R}}$. Furthermore a complex affine structure of $\tilde{\mathfrak{b}}$ induces a real affine structure of $\tilde{\mathfrak{b}}^{\mathbf{R}}$. So, in case of $\mathfrak{g}=\tilde{\mathfrak{g}}^{\mathbf{R}}$, (2) follows from (1).

We thank to Professor M. Goto for his suggestion to the real case. Our original paper dealt only with a class of maximal solvable subalgebras associated

* Present address: Ōita Institute of Technology, Tanoo, Ichiki, Ōita, Japan.

with the Iwasawa decomposition of $\mathfrak{g}$, which are similar to Borel subalgebras. We also thank to Professor H. Matsumoto who informed us of his result on the conjugacy classes of maximal solvable subalgebras of real semi-simple Lie algebras [3].

2. Proof of (1). Let $\tilde{\mathfrak{g}}$ be a complex semi-simple Lie algebra, $\tilde{\mathfrak{h}}$ a Cartan subalgebra of $\tilde{\mathfrak{g}}$ and $\Delta=\Delta(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$ the set of all non-zero roots with respect to $\tilde{\mathfrak{h}}$. As usual, we introduce a lexicographic order and let Δ^+ (resp. $\Pi=\{\alpha_1,\dots,\alpha_l\}$) be the set of all positive roots (resp. the fundamental system of roots) with respect to this order. Put $\tilde{\mathfrak{n}}^+=\sum_{\alpha\in\Delta^+}\tilde{\mathfrak{g}}^\alpha$, where $\tilde{\mathfrak{g}}^\alpha$ denotes the root space corresponding to $\alpha\in\Delta^+$. Then $\tilde{\mathfrak{b}}=\tilde{\mathfrak{n}}^+ +\tilde{\mathfrak{h}}$ is called a Borel subalgebra. Every maximal solvable subalgebra of $\tilde{\mathfrak{g}}$ is conjugate, under an inner automorphism of $\tilde{\mathfrak{g}}$, to the above standard $\tilde{\mathfrak{b}}$ (cf. [2]). So it is sufficient to show that the above $\tilde{\mathfrak{b}}$ admits an affine structure.

Each $\alpha\in\Delta^+$ is written uniquely as $\alpha=\sum_{k=1}^{l} m_k\alpha_k$ ($m_k\geqq 0$, integers). Now we define a gradation of $\tilde{\mathfrak{n}}^+$ by setting $\tilde{\mathfrak{n}}_i^+=\sum_{|\alpha|=i}\tilde{\mathfrak{g}}^\alpha$, where we put $|\alpha|=\sum_{k=1}^{l} m_k$ for $\alpha=\sum_{k=1}^{l} m_k\alpha_k\in\Delta^+$. Let $\mathbf{I}$ be the set $\{i=\sum_{k=1}^{l} m_k;\ \sum_{k=1}^{l} m_k\alpha_k\in\Delta^+\}$. Then $\{\tilde{\mathfrak{n}}_i^+;\ i\in\mathbf{I}\}$ gives a gradation of $\tilde{\mathfrak{n}}^+$ by positive integers and $\tilde{\mathfrak{h}}$ preserves the gradation, i.e., $\tilde{\mathfrak{n}}^+=\sum_{i\in\mathbf{I}}\tilde{\mathfrak{n}}_i^+$ (direct sum), $[\tilde{\mathfrak{n}}_i^+, \tilde{\mathfrak{n}}_j^+]\subseteq\tilde{\mathfrak{n}}_{i+j}^+$ and $\mathrm{ad}(\tilde{\mathfrak{h}})\tilde{\mathfrak{n}}_i^+\subseteq\tilde{\mathfrak{n}}_i^+$. Therefore $\tilde{\mathfrak{b}}=\tilde{\mathfrak{n}}^+ +\tilde{\mathfrak{h}}$ satisfies the properties stated in §1 of [7]. Consequently $\tilde{\mathfrak{b}}$ admits a complex affine structure by the theorem of [7]. As a real algebra, $\tilde{\mathfrak{b}}^{\mathbf{R}}$ admits a real affine structure (cf. [7]). Thus the proof of (1) is completed.

3. Proof of (2). Let $\mathfrak{g}$ be a non-compact real semi-simple Lie algebra and $\mathfrak{s}$ a maximal solvable subalgebra of $\mathfrak{g}$. Contrary to the complex case, there are finitely many conjugacy classes of maximal solvable subalgebras of $\mathfrak{g}$ and they were classified completely by H. Matsumoto [3].

Let $\mathfrak{g}_{\mathbf{C}}$ be the complexification of $\mathfrak{g}$. An element X of $\mathfrak{g}$ is said to be *nilpotent* (resp. *semi-simple, real semi-simple*) if $\mathrm{ad}\,X$ is an endomorphism of $\mathfrak{g}_{\mathbf{C}}$ which is nilpotent (resp. semi-simple, semi-simple with real eigen-values). Let H be a real semi-simple element of $\mathfrak{g}$. We denote by $\mathfrak{g}_0(H)$, $\mathfrak{g}_+(H)$ and $\mathfrak{g}_*(H)$ the sums of subspaces of $\mathfrak{g}$ corresponding to zero, positive and non-negative eigen-values of $\mathrm{ad}\,H$ respectively. For a subspace $\mathfrak{l}$ of $\mathfrak{g}$, $\mathfrak{l}^\perp$ denotes the orthogonal subspace of $\mathfrak{l}$ with respect to the Killing form of $\mathfrak{g}$. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be a Cartan decomposition, $\mathfrak{h}^-$ a maximal abelian subspace of $\mathfrak{p}$ and $\mathscr{C}$ the positive Weyl chamber of $\mathfrak{h}^-$ with respect to Π^-:

$$\mathscr{C}=\{H;\ H\in\mathfrak{h}^-,\ \langle\gamma_i, H\rangle\geqq 0\quad\text{for any}\quad \gamma_i\in\Pi^-\}.$$

For the precise definition of Π^-, see [3].

Under the above situation, we summarize here some results in [3] which are necessary for our later argument.

(a) *Let $\mathfrak{m}$ be a parabolic subalgebra of $\mathfrak{g}$, i.e., $\mathfrak{m}_{\mathbf{C}}$ contains a Borel subalgebra of $\mathfrak{g}_{\mathbf{C}}$. Then $\mathfrak{m}=\mathfrak{g}_*(H)$ for some real semi-simple element H and conversely $\mathfrak{g}_*(H)$ is parabolic for any real semi-simple element H.*

(b) *Every real semi-simple element of $\mathfrak{g}$ is conjugate to some element of $\mathscr{C}$. Therefore every parabolic subalgebra is conjugate to some $\mathfrak{g}_*(H)$ $(H\in\mathscr{C})$ under an inner automorphism of $\mathfrak{g}$.*

(c) *Let $\mathfrak{s}$ be a maximal solvable subalgebra of $\mathfrak{g}$, $\mathfrak{n}$ its ideal composed of nilpotent elements and $\mathfrak{m}$ the normalizer of $\mathfrak{n}$ in $\mathfrak{g}$. Then $\mathfrak{m}=\mathfrak{n}^\perp$ and $\mathfrak{m}$ is the smallest parabolic subalgebra of $\mathfrak{g}$ which contains $\mathfrak{s}$. Conversely let $\mathfrak{m}$ be a parabolic subalgebra and $\mathfrak{s}$ a maximal solvable subalgebra of $\mathfrak{m}$. Then $\mathfrak{s}$ is also maximal in $\mathfrak{g}$.*

Now we shall prove (2). Let $\mathfrak{g}$ and $\mathfrak{s}$ be as in (2) of Theorem and $\mathfrak{m}$ the smallest parabolic subalgebra which contains $\mathfrak{s}$. By virtue of (b), we can assume, without loss of generality, $\mathfrak{m}=\mathfrak{g}_*(H)\supset\mathfrak{s}$ $(H\in\mathscr{C})$. Let $\mathfrak{r}$ be the radical of $\mathfrak{m}$ and $\mathfrak{g}'$ a maximal semi-simple subalgebra of $\mathfrak{m}$: $\mathfrak{m}=\mathfrak{r}+\mathfrak{g}'$. Then $\mathfrak{s}=\mathfrak{r}+\mathfrak{s}\cap\mathfrak{g}'$ and $\mathfrak{s}\cap\mathfrak{g}'$ is a Cartan subalgebra of $\mathfrak{g}'$ (cf. [3]).

First assume $\mathfrak{n}=(0)$, where $\mathfrak{n}$ is the ideal composed of nilpotent elements of $\mathfrak{s}$. Then by (c), $\mathfrak{m}=\mathfrak{n}^\perp=\mathfrak{g}$. It follows that $\mathfrak{r}=(0)$, $\mathfrak{g}'=\mathfrak{g}$ and $\mathfrak{s}\cap\mathfrak{g}'=\mathfrak{s}\cap\mathfrak{g}=\mathfrak{s}$ is a Cartan subalgebra of $\mathfrak{g}'=\mathfrak{g}$. Therefore $\mathfrak{s}$ is abelian and consequently $\mathfrak{s}$ admits an affine structure. Precisely speaking, $\mathfrak{s}$ is a *compact* Cartan subalgebra in this case (cf. [3]).

Next assume $\mathfrak{n}\neq(0)$. Then $\mathfrak{m}\neq\mathfrak{g}$ by (c) and as mentioned in the proof of Lemma 3.1 in [3], $\mathfrak{s}$ possesses a non-zero real semi-simple element. We may assume that H is in $\mathscr{C}$. Then $\mathfrak{g}_0(H)$ is reductive. So $\mathfrak{g}_0(H)=\mathfrak{c}_0+\mathfrak{g}_0'$, where $\mathfrak{c}_0$ is the center of $\mathfrak{g}_0(H)$ and $\mathfrak{g}_0'$ is the derived algebra of $\mathfrak{g}_0(H)$ which is semi-simple. Since $\mathfrak{s}$ contains the radical $\mathfrak{r}$ of $\mathfrak{m}$, $\mathfrak{r}$ contains the nilpotent ideal $\mathfrak{g}_+(H)$ of $\mathfrak{m}$ and $\mathfrak{s}$ is a maximal solvable subalgebra of $\mathfrak{m}$, it is easy to see that $\mathfrak{s}=\mathfrak{c}_0+\mathfrak{s}_0+\mathfrak{g}_+(H)$ for some maximal solvable subalgebra $\mathfrak{s}_0$ of $\mathfrak{g}_0'$ (and conversely any maximal solvable subalgebra of $\mathfrak{m}$ is of this form). Now we shall show by induction on

$\dim \mathfrak{g}$ that $\mathfrak{s}$ admits an affine structure. If $\dim \mathfrak{g}=3$, then $\dim \mathfrak{s}=1$ or 2. In this case $\mathfrak{s}$ admits an affine structure (cf. [5]). Assume (2) is true for any $\mathfrak{g}$ and for any maximal solvable subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ such that $\dim \mathfrak{g} \leqq n_0$. Let $n\ (>n_0)$ be the least positive integer such that there exists a non-compact real semi-simple Lie algebra $\mathfrak{g}$ whose dimension is equal to n. Let $\mathfrak{g}$ be any one of such algebras, that is, $\mathfrak{g}$ such that $\dim \mathfrak{g}=n$, and $\mathfrak{s}$ any maximal solvable subalgebra of $\mathfrak{g}$. We may assume without loss of generality that $\mathfrak{n} \neq (0)$. Then as mentioned above we have $\mathfrak{s}=\mathfrak{c}_0+\mathfrak{s}_0+\mathfrak{g}_+(H)$. Since $\mathfrak{n}\neq(0)$, it follows that $\mathfrak{g}_+(H)\neq(0)$ and $\dim \mathfrak{g}_0' \leqq \dim \mathfrak{g}-\dim \mathfrak{g}_+(H)<\dim \mathfrak{g}$. Then by induction assumption, $\mathfrak{s}_0$ admits an affine structure, that is, there exists an affine representation $\rho_{\mathfrak{s}_0}\colon \mathfrak{s}_0 \to \mathfrak{a}(q)$ such that the analytic subgroup $\mathbf{G}(q)$ of $\mathbf{A}(q)$ operates simply transitively on $\mathbf{R}^q$, where $\mathbf{A}(q)$ is the affine transformation group of $\mathbf{R}^q$, $\mathfrak{a}(q)$ is its Lie algebra and $q=\dim \mathfrak{s}_0$. With respect to a suitable basis of $\mathfrak{s}_0$, $\rho_{\mathfrak{s}_0}(Y)$ $(Y\in\mathfrak{s}_0)$ is represented by the following matrix:

$$\rho_{\mathfrak{s}_0}(Y)=\left(\begin{array}{c|c} A_0(Y) & \mathbf{v}(Y) \\ \hline 0 & 0 \end{array}\right),$$

where $A_0(Y)\in \mathfrak{gl}(q,\mathbf{R})$ and $\mathbf{v}(Y)={}^t(v_1(Y),\dots,v_q(Y))\in\mathbf{R}^q$. Put $D=\operatorname{ad} H$. Then $D|_{\mathfrak{n}^+}$ is a non-singular derivation, which we express again by D. So, by the result due to Scheuneman [4], $\mathfrak{n}^+\equiv\mathfrak{g}_+(H)$ admits an affine structure, that is, there exists an affine representation $\rho_{\mathfrak{n}^+}\colon \mathfrak{n}^+\to\mathfrak{a}(r)$ $(r=\dim\mathfrak{n}^+)$ which satisfies the same property as $\rho_{\mathfrak{s}_0}$. With respect to a suitable basis of $\mathfrak{n}^+$ $\rho_{\mathfrak{n}^+}(Z)$ $(Z\in\mathfrak{n}^+)$ is represented by the following matrix:

$$\rho_{\mathfrak{n}^+}(Z)=\left(\begin{array}{c|c} \operatorname{ad}_{\mathfrak{n}^+}Z & DZ \\ \hline 0 & 0 \end{array}\right),$$

where $\mathbf{R}^r$ is identified with $\mathfrak{n}^+$. Using $\rho_{\mathfrak{s}_0}$ and $\rho_{\mathfrak{n}^+}$, we construct an affine representation $\rho\colon \mathfrak{s}\to\mathfrak{a}(s)$ $(s=\dim\mathfrak{s})$ which gives an affine structure of $\mathfrak{s}$. Choose an arbitrary basis $\{X_1,\dots,X_p\}$ of $\mathfrak{c}_0$ and use the above bases of $\mathfrak{s}_0$ and $\mathfrak{n}^+$. With respect to this basis of $\mathfrak{s}$, define ρ by

$$\rho(X)=\left(\begin{array}{c|c|c|c} \overbrace{0}^{p} & \overbrace{0}^{q} & \overbrace{0}^{r} & X \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \operatorname{ad}X|_{\mathfrak{n}^+} & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right) \quad (X\in\mathfrak{c}_0,\ p=\dim\mathfrak{c}_0),$$

$$\rho(Y)=\left(\begin{array}{c|c|c|c} 0 & 0 & 0 & \\ \hline 0 & A_0(Y) & 0 & \mathbf{v}(Y) \\ \hline 0 & 0 & \operatorname{ad}Y|_{\mathfrak{n}^+} & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right) \quad (Y\in\mathfrak{s}_0),$$

$$\rho(Z)=\left(\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \operatorname{ad}_{\mathfrak{n}^+}Z & DZ \\ \hline 0 & 0 & 0 & 0 \end{array}\right) \quad (Z\in\mathfrak{n}^+=\mathfrak{g}_+(H))$$

and extend ρ by linearity. Then from the shape of ρ and from the fact that $\rho_{\mathfrak{s}_0}$, $\rho_{\mathfrak{n}^+}$ are affine structures of $\mathfrak{s}_0$, $\mathfrak{n}^+$ respectively and the derivation $D=\operatorname{ad}H$ is zero on $\mathfrak{c}_0+\mathfrak{s}_0$, it follows that ρ is a faithful representation and the set of all translation parts of $\rho(\mathfrak{s})$ coincides with the whole $\mathbf{R}^s$ (cf. [7]). Finally we have to show that the analytic subgroup $\mathbf{G}(s)$ with Lie algebra $\rho(\mathfrak{s})$ of the affine transformation group $\mathbf{A}(s)$ operates transitively on $\mathbf{R}^s$. Then the simplicity of its operation follows (cf. [1], [5], [6]). Let $\mathbf{v}={}^t(a_1,\dots,a_p,b_1,\dots,b_q,c_1,\dots,c_r,1)$ by any point of $\mathbf{R}^s\equiv\mathbf{R}^s\times 1\subset\mathbf{R}^{s+1}$ and denote ${}^t(\underbrace{0,\dots,0}_{p},\underbrace{0,\dots,0}_{q},\underbrace{0,\dots,0}_{r},1)$ by $\mathbf{0}$. Then we have

$$\tilde{g}_1\cdot\mathbf{0}={}^t(a_1,\dots,a_p,0,\dots,0,0,\dots,0,1),$$

where we put $\tilde{g}_1=\prod_{k=p}^{1}\exp a_k\rho(X_k)$. Since $\rho_{\mathfrak{s}_0}$ and $\rho_{\mathfrak{n}^+}$ give affine structures of $\mathfrak{s}_0$ and $\mathfrak{n}^+$ respectively, there exist $g_2\in\mathbf{G}(q)$ and $g_3\in\mathbf{G}(r)$ such that

$$g_2\cdot{}^t(0,\dots,0,1)={}^t(b_1,\dots,b_q,1),$$

$$g_3\cdot{}^t(0,\dots,0,1)={}^t(c_1,\dots,c_r,1),$$

where $g_2=\prod_{k=1}^{q}\exp\rho_{\mathfrak{s}_0}(Y_k)$ for some $Y_k\in\mathfrak{s}_0$ and $g_3=\prod_{k=1}^{r}\exp\rho_{\mathfrak{n}^+}(Z_k)$ for some $Z_k\in\mathfrak{n}^+$. We put $\tilde{g}_2=\prod_{k=1}^{q}\exp\rho(Y_k)$ and $\tilde{g}_3=\prod_{k=1}^{r}\exp\rho(Z_k)$. Then from the shape of ρ, it follows that

$$\tilde{g}_3\cdot\tilde{g}_2\cdot\tilde{g}_1\cdot\mathbf{0}=\mathbf{v}.$$

This implies the transitivity of $\mathbf{G}(s)$. Summing up, $\rho\colon\mathfrak{s}\to\mathfrak{a}(s)$ gives an affine structure of $\mathfrak{s}$. Thus the induction is completed and (2) is proved.

References

[1] L. AUSLANDER, *Simply transitive groups of affine motions*, Amer. J. Math., **99** (1977), 809–826.
[2] A. BOREL, *Groupes linéaires algébriques*, Ann. of Math., **64** (1956), 20–82.
[3] H. MATSUMOTO, *Quelques remarques sur les groupes de Lie algébriques réels*, J. Math. Soc. Japan **16** (1964), 419–446.
[4] J. SCHEUNEMAN, *Examples of compact locally affine spaces*, Bull. Amer. Math. Soc., **77** (1971), 589–592.
[5] S. YAMAGUCHI, *On complete affinely flat structures of some solvable Lie groups*, Mem. Fac. Sci. Kyushu Univ., **33** (1979), 209–218.
[6] S. YAMAGUCHI, *Supplement to "On complete affinely flat structures .." and an application to nilpotent Lie groups of dimension* $\leqq 5$, ibid., 219–223.
[7] S. YAMAGUCHI, *Affine structure of some solvable Lie group*, Mem. Fac. Sci. Kyushu Univ., **34** (1980), 339–345.