ベキ零リー環の分類について

竹　本　義　夫

On Classification of Nilpotent Lie Algebra

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日本文理大学紀要
第11巻　第1号
昭和57年10月

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Abstract

Introducing a maximal nilpotent Lie algebra for a given rank and step, and looking upon a nilpotent Lie algebra as a subalgebra of it, we are able to characterize the nilpotent Lie algebras as linear relations among the basis of the maximal Lie algebra and classify the nilpotent Lie algebras by using this characterization.

At last, we show this method in the case of 4-step nilpotent Lie algebra of rank 2.

§1. Introduction

Regarding the classification of nilpotent Lie algebras, only a few studies have been done. G. Vrănceanu classified 6-dimentional nilpotent Lie algebras (cf. [1], [3]), and S. Yamaguchi constructed many examples for more than 7-dimentional ones (cf. [4]).

In this paper, we find an another method for constructing and classifying the nilpotent Lie algebras by means of the basis which was used in [2].

§2. The basis of $\mathfrak{g}$

In this section, we shall suitably rearrange a basis of nilpotent Lie algebras and investigate the rules among the basis

Let $\mathfrak{g}$ be a k-step nilpotent Lie algebra over **R**. That is, $\mathfrak{g}$ is a vector space over **R** and there is given a bilinear form $[\ ,\ ]$ on $\mathfrak{g}$ satisfying

1) $[X, Y]+[Y, X]=0$ for $X, Y \in \mathfrak{g}$,
2) Jacobi identity,
3) $\mathfrak{g}^{(k)}=0$, where $\mathfrak{g}^{(h)}\ (2 \leq h \leq k)$ is defined by $\mathfrak{g}^{(h)}=[\mathfrak{g}, \mathfrak{g}^{(h-1)}]$ inductively and

*Received July 29, 1982 **Instructor of mathematics

$\mathfrak{g}^{(1)}=[\mathfrak{g}, \mathfrak{g}]$.

Let $x_{11}, x_{12}, \cdots\cdots, x_{1n_1}$ be representatives of the basis of $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$, where $n_1=\dim(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])$, then

$$\mathfrak{g}=\sum_{i=1}^{n_1} \mathbf{R}x_{1i}+[\mathfrak{g}, \mathfrak{g}].$$

Moreover

$$[\mathfrak{g}, \mathfrak{g}]=\sum_{i=1}^{n_1}\left(\sum_{j=1}^{n_1} \mathbf{R}[x_{1i}, x_{1j}]\right)+[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]].$$

We choose then maximal family of independent vectors mod $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]$ in $\{[x_{1i}, x_{1j}]\ ;\ i, j=1, 2, \cdots\cdots, n_1\}$ and denote them by $x_{21}, x_{22}, \cdots\cdots, x_{2n_2}$, where $n_2=\dim([\mathfrak{g}, \mathfrak{g}]/[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]])$, then

$$[\mathfrak{g}, \mathfrak{g}]=\sum_{j=1}^{n_2} \mathbf{R}x_{2j}+[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]$$

Moreover

$$[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]=\sum_{i=1}^{n_1}\left(\sum_{j=1}^{n_2} \mathbf{R}[x_{1i}, x_{2j}]\right)+[\mathfrak{g}, [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]].$$

We choose then a maximal family of independent vectors mod $[\mathfrak{g}, [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]]$ of $([x_{1i}, x_{2j}]\ ;\ i=1, 2, \cdots\cdots, n_1, j=1, 2, \cdots\cdots, n_2)$ and denote them by $x_{31}, x_{32}, \cdots\cdots, x_{3n_3}$, where $n_3=\dim([\mathfrak{g}, \mathfrak{g}]/[\mathfrak{g}, [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]])$.

Repeating the above procedure we can choose a maximal family of independent vectors mod $\mathfrak{g}^{(h)}$ of $\{[x_{1i}, x_{(h-1)j}]\ ;\ i=1, 2, \cdots\cdots, n_1, j=1, 2, \cdots\cdots, n_{h-1}\}$ for any $h\ (1 \leq h \leq k)$ and denote them by $x_{h1}, \cdots\cdots, x_{hn_h}$, where $n_h=\dim(\mathfrak{g}^{(h-1)}/\mathfrak{g}^{(h)})$.

Definition 1. *For a given positive integer* n, *we may consider all* k-*step nilpotent Lie algebras of rank* n (*i. e.*, $\dim(\mathfrak{g}/\mathfrak{g}^{(1)})=n$).

Then, we can choose among these such one as $N=n_1+\cdots+n_k$, where $n_1=n$, *is maximal in the above procedure.*

We call it the k-*step maximal nilpotent Lie algebra of rank* n.

Definition 2. *Let* $\mathfrak{g}$ *be a nilpotent Lie algebra and* $x_1, x_2, \cdots\cdots, x_n$ *representatives of the basis of* $\mathfrak{g}/\mathfrak{g}^{(1)}$, then

$\mathfrak{g}=\mathfrak{g}_1+\mathfrak{g}_2+\cdots\cdots+\mathfrak{g}_k$, *where* $\mathfrak{g}_h\ (h=1, 2, \cdots\cdot, k)$ *is a vector space spanned by* $\{x_{hi}\ ;\ i=1, 2, \cdots\cdot n_h\}$ *and we call these vectors* $x_1, x_2, \cdots\cdot, x_n$ *generators*.

Let $\bar{\mathfrak{g}}$ be a k-step maximal nilpotent Lie algebra of rank n and $\mathfrak{M}=\bigcup_{h=1}^{k} \mathfrak{M}_h$, where $\mathfrak{M}_h=\{[x_{i_h}, [x_{i_{h-1}}, \cdots\cdots, [x_{i_2}, x_{i_1}]\cdots\cdot]]\ ;\ i_1, i_2, \cdots\cdot\ i_h=1, 2, \cdots, n\}$, then the following relations hold :

1) $[X, Y]+[Y, X]=0 \qquad (X, Y \in \mathfrak{M})$,
2) Jacobi identity, i. e.,

$$[X, [Y, Z]]+[Z, [X, Y]]+[Y, [Z, X]]=0 \qquad (X, Y, Z \in \mathfrak{M}),$$

and we can choose the basis of $\mathfrak{g}$ from $\mathfrak{M}$ by means of these relations.

Further details are as follows :

Hereafter we use the following notations for simplicity :

$$x_{i_h} x_{i_{h-1}} \cdots\cdots x_{i_1} X = \mathrm{ad}x_{i_h}\, \mathrm{ad}x_{i_{h-1}} \cdots\cdots \mathrm{ad}x_{i_1} X$$

for $x_{i_h} x_{i_{h-1}} \cdots x_{i_1} \in \mathfrak{M}_h$ $(1 \leq h \leq k)$, $X \in \mathfrak{g}$.

Lemma 1. *The relations* 1), 2) *are equivalent to the following relations* $1)_1$ *and* $2)_1$:

$$1)_1 \quad x_l Y = -[Y, x_l] \qquad (x_l \in \mathfrak{M}_1,\ Y \in \mathfrak{M}),$$

$$2)_1 \quad [x_l Y, Z] = x_l [Y, Z] - [Y, x_l Z] \quad (x_l \in \mathfrak{M}_1,\ Y, Z \in \mathfrak{M}).$$

Proof.

1), 2) $\Rightarrow$ $1)_1$, $2)_1$ are clear.

Conversely for showing $1)_1$, $2)_1$ $\Rightarrow$ 1), 2), we shall prove the following relations $1)_h$, $2)_h$ $(1 \leq h \leq k)$ by an inductive method :

$$1)_h \quad [X, Y] = -[Y, X] \qquad (X \in \mathfrak{M}_h,\ Y \in \mathfrak{M})$$

$$2)_h \quad [[X, Y], Z] = [X, [Y, Z]] - [Y, [X, Z]] \qquad (X \in \mathfrak{M}_h,\ Y, Z \in \mathfrak{M})$$

For $X = x_l \in \mathfrak{M}_1$, then they are $1)_1$, $2)_1$ themselvs.

For $X \in \mathfrak{M}_h$ $(1 \leq h < k)$, we assume that $1)_h$ $2)_h$ are true, then

$$\begin{aligned}
1)_{h+1}:\ [x_l X, Y] &= [x_l, [X, Y]] - [X, x_l Y] && \because 2)_1\\
&= [x_l, [X, Y]] + [x_l Y, X] && \because 1)_h\\
&= [x_l, [X, Y]] + [x_l, [Y, X]] - [Y, x_l X] && \because 2)_1\\
&= -[Y, x_l X] && \because 1)_h
\end{aligned}$$

$$(x_l \in \mathfrak{M}_1,\ X \in \mathfrak{M}_h,\ Y \in \mathfrak{M}).$$

$$\begin{aligned}
2)_{h+1}:\ [[x_l X, Y], Z] &= [x_l [X, Y] - [X, x_l Y], Z] && \because 2)_1\\
&= [x_l [X, Y], Z] - [[X, x_l Y], Z]\\
&= (x_l [[X, Y], Z] - [[X, Y], x_l Z]) && \because 2)_1\\
&\quad - ([X, [x_l Y, Z]] - [x_l Y, [X, Z]]) && \because 2)_h\\
&= (x_l [X, [Y, Z]] - x_l [Y, [X, Z]]) && \because 2)_h\\
&\quad - ([X, [Y, x_l Z]] - [Y, [X, x_l Z]]) && \because 2)_h\\
&\quad - ([X, x_l [Y, Z]] - [X, [Y, x_l Z]]) && \because 2)_1\\
&\quad + (x_l [Y, [X, Z]] - [Y, x_l [X, Z]]) && \because 2)_1\\
&= (x_l [X, [Y, Z]] - [X, x_l [Y, Z]])\\
&\quad - ([Y, x_l [X, Z]] - [Y, [X, x_l Z]])\\
&= [x_l X, [Y, Z]] - [Y, [x_l X, Z]] && \because 2)_1
\end{aligned}$$

$$(x_l \in \mathfrak{M}_1,\ X \in \mathfrak{M}_h,\ Y, Z \in \mathfrak{M}).$$

Q.E.D.

Lemma 2. *The relation* $2)_1$ *is equivalent to the following relation* $2)'_1$:

$$2)'_1 \quad [x_{i_h} x_{i_{h-1}} \cdots\cdots x_{i_1}, Z] = \sum_{\varepsilon_h, \cdots, \varepsilon_2 = 0,1} (-1)^{\varepsilon_h + \cdots + \varepsilon_2} x_{i_h}^{1-\varepsilon_h} \cdots x_{i_2}^{1-\varepsilon_2} x_{i_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_h}^{\varepsilon_h} Z$$

$$(x_{i_h} \cdots\cdots x_{i_1} \in \mathfrak{M}_h,\ Z \in \mathfrak{M})$$

where $x_l^1 X = x_l X$, $x_l^0 X = X$ $\quad (x_l \in \mathfrak{M}_1,\ X \in \mathfrak{M})$.

This means that the bracket of two elements of $\mathfrak{M}$ *is represented uniquely by a linear combination among elements of* $\mathfrak{M}$.

Proof.

Using the relation $2)_1$, we get

$$\begin{aligned}
&[x_{i_h} x_{i_{h-1}} \cdots\cdots x_{i_1}, Z] && \because 2)_1\\
&= x_{i_h} [x_{i_{h-1}} \cdots x_{i_1}, Z] - [x_{i_{h-1}} \cdots x_{i_1}, x_{i_h} Z] && \because 2)_1\\
&= x_{i_h} x_{i_{h-1}} [x_{i_{h-2}} \cdots x_{i_1}, Z] - x_{i_h} [x_{i_{h-2}} \cdots x_{i_1}, x_{i_{h-1}} Z] && \because 2)_1\\
&\quad - x_{i_{h-1}} [x_{i_{h-2}} \cdots x_{i_1}, x_{i_h} Z] + [x_{i_{h-2}} \cdots x_{i_1}, x_{i_{h-1}} x_{i_h} Z].
\end{aligned}$$

Repeating this deformation, we get $2)'_1$.

Conversely we assume that the above relations hold. Then

$$\begin{aligned}
[x_l Y, Z] &= [x_l x_{i_h} \cdots\cdots x_{i_1}, Z] && \because 2)_1\\
&= \sum_{\varepsilon, \varepsilon_h, \cdots, \varepsilon_2 = 0,1} (-1)^{\varepsilon + \varepsilon_h + \cdots + \varepsilon_2} x_l^{1-\varepsilon} x_{i_h}^{1-\varepsilon_h} \cdots x_{i_1} \cdots x_{i_h}^{\varepsilon_h} x_l^{\varepsilon} Z\\
&= x_l \sum_{\varepsilon_h, \cdots, \varepsilon_2 = 0,1} (-1)^{\varepsilon_h + \cdots + \varepsilon_2} x_{i_h}^{1-\varepsilon_h} \cdots x_{i_1} \cdots x_{i_h}^{\varepsilon_h} Z\\
&\quad - \sum_{\varepsilon_h, \cdots, \varepsilon_2 = 0,1} (-1)^{\varepsilon_h + \cdots + \varepsilon_2} x_{i_h}^{1-\varepsilon_h} \cdots x_{i_1} \cdots x_{i_h}^{\varepsilon_h} x_l Z && \because 2)_1\\
&= x_l [Y, Z] - [Y, x_l Z].
\end{aligned}$$

Hence there are some relations on the elements of $\mathfrak{M}_h$ $(1 \leq h \leq k)$:

$$\begin{aligned}
1)'_1:\quad x_l x_{i_{h-1}} \cdots x_{i_1} &= -[x_{i_h} \cdots x_{i_1}, x_l] && \because 1)_1\\
&= -\sum_{\varepsilon_h, \cdots, \varepsilon_2 = 0,1} (-1)^{\varepsilon_h + \cdots + \varepsilon_2} x_{i_h}^{1-\varepsilon_h} \cdots x_{i_1} \cdots x_{i_h}^{\varepsilon_h} x_l && \because 2)'_1
\end{aligned}$$

$$(x_l \in \mathfrak{M}_1,\ x_{i_h}, \cdots, x_{i_1} \in \mathfrak{M}_h).$$

Q.E.D.

§3. Maximal nilpotent Lie algelra

In this section, we shall find a maximal nilpotent Lie algebra $\bar{\mathfrak{g}}$ which contains all nilpotent Lie algebras $\mathfrak{g}$ with the same rank and step (as subalgebras) and investigate the relation among these.

Using the above relation $1)'_1$, we can choose a maximal family of linearly independent vectors (i.e., a basis of $\bar{\mathfrak{g}}$) from $\mathfrak{M}$ as follows:

a) We introduce a lexicographic order in $\mathfrak{M}$.

i) $x_i < x_j$ $(x_i, x_j \in \mathfrak{M}_1)$ if and only if $i < j$,

ii) $x_{i_h} x_{i_{h-1}} \cdots x_{i_1} < x_{j_h} x_{j_{h-1}} \cdots x_{j_1}$ $(x_{i_h} \cdots x_{i_1},\ x_{j_h} \cdots x_{j_1} \in \mathfrak{M}_h)$ if and only if there is such a h_0 $(1 \leqq h_0 \leqq h)$ that $x_{i_h} < x_{j_h}, \cdots, x_{i_{h_0}} < x_{j_{h_0}}$, $x_{i_{h_0+1}} = x_{j_{h_0+1}}, \cdots, x_{i_1} = x_{j_1}$,

iii) $X < Y$ $(X \in \mathfrak{M}_h,\ Y \in \mathfrak{M}_{h'})$ if and only if $h < h'$.

b) We adopt the above order and choose a maximal family of linearly independent vectors from each $\mathfrak{M}_h$ $(1 \leqq h \leqq k)$ inductively.

i) All vectors in $\widehat{\mathfrak{M}_1}$, i.e., x_1, $x_2, \cdots$, x_n are linearly independent.

ii) Let Y_1, $Y_2, \cdots$, Y_{n_h} be a maximal family of linearly independent vectors in $\mathfrak{M}_h$ and $Y_i < Y_j$ when $i < j$.

We put here

$$\begin{aligned} \mathfrak{M}'_{h+1} = \{ & x_n Y_{n_h},\ x_{n-1} Y_{n_h}, \cdots\cdots,\ x_1 Y_{n_h} \\ & x_n Y_{n_h-1},\ x_{n-1} Y_{n_h-1}, \cdots, x_1 Y_{n_h-1} \\ & \vdots \\ & x_n Y_1,\ x_{n-1} Y_1, \cdots\cdots, x_1 Y_1 \} \subset \mathfrak{M}_{h+1} \end{aligned}$$

Then all the elements in $\mathfrak{M}_{h+1}$ depend upon some elements in $\mathfrak{M}'_{h+1}$ and we can choose maximal family of linearly independent vectors in $\mathfrak{M}'_{h+1}$ through the following steps:

We first express the highest element $x_n Y_{n_h} = -[Y_{n_h}, x_n]$ in $\mathfrak{M}'_{h+1}$ by the linear combination of other elements in $\mathfrak{M}'_{h+1}$ by the relation $1)'_1$. Then if this relation is not an identity, the element which is the lowest element in the terms of the relation, can be excluded and we then denote for simplicity this set the same notation $\mathfrak{M}_{h+1}$.

Secondly the next element $x_{n-1} Y_{n_h}$ in $\mathfrak{M}'_{h+1}$ is also represented by another linear combination of elements in $\mathfrak{M}'_{h+1}$ and if this relation is not an identity, then we denote the set which excludes the lowest element in this relation the same notation $\mathfrak{M}'_{h+1}$.

We further carry out the same procedure likewise for all the other linearly independent vectors. Consequently all the elements in $\mathfrak{M}'_{h+1}$ of the last stage, are maximal family of linearly independent vectors in $\mathfrak{M}'_{h+1}$ and other elements depend upon these elements in $\mathfrak{M}'_{h+1}$.

Definition 3. *We call a vector* Y *in* $\mathfrak{M}_{h'}$ *the descended vector* of $X \in \mathfrak{M}_h$ $(h < h')$, *if there are vectors* $x_{h'}$, $x_{h'-1} \cdots x_{h+1}$ *in* $\mathfrak{M}_1$ *such that* $Y = x_{h'} x_{h'-1} \cdots x_{h+1} X$.

Theorem 1. i) *Maximal nilpotent Lie algebras with the same rank and step are all isomorphic.*
ii) *Nilpotent Lie algebra* $\mathfrak{g}$ *of rank* n *is a subalgebra of the maximal nilpotent Lie algebra* $\bar{\mathfrak{g}}$ *of the same rank. That is, the basis of* $\mathfrak{g}$ *can be obtained by linear combinations of some vectors X in the basis of* $\bar{\mathfrak{g}}$ *and higher vectors which are not the descended vector of X. Moreover we can choose above vectors X as is not dependent upon the higher vectors than itself.*

Conversely when we give some linear combinations among the basis of $\bar{\mathfrak{g}}$ *as stated above, then we obtain a nilpotent Lie subalgebra of* $\bar{\mathfrak{g}}$.

Proof.

The basis of maximal nilpotent Lie algebra can be obtained from $\mathfrak{M}$ through the only relation $1)'_1$. Therefore we get i).

The basis of $\mathfrak{g}_{h+1}$ is chosen from $\mathfrak{M}'_{h+1} = \{x_i Y_j ; x_i \in \mathfrak{M}_1,\ Y_j\ (j=1, \cdots, n_h)$ is a basis of $\mathfrak{g}_h\}$. Therefore if a vector X depends upon higher vectors or equals to 0 mod $\mathfrak{g}^{(h+1)}$, then the desended vectors of X can be excluded from basis of $\mathfrak{g}$.

Conversely the Lie algebra stated in ii) is well defined and nilpotent. Therefore we get ii).
Q. E. D.

§ 4. Example

In this section, we shall apply the method stated in Sec. 1 and 2 to the case of 2-rank, 6-step and 3-rank, 4-step nilpotent Lie algebras and get the maximal ones of them.

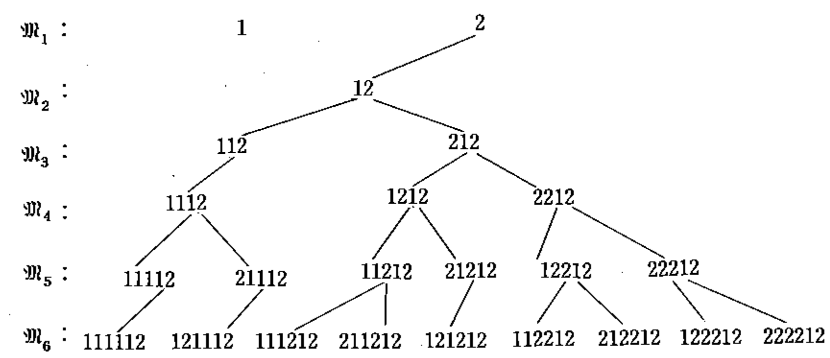
Hereafter we use the following notation for simplicity:

$$a * i_h i_{h+1} \cdots i_1 = a * x_{i_h} x_{i_{h-1}} \cdots x_{i_1} \in \mathfrak{M}_h \ (a \in \mathbf{R},\ 1 \leq h \leq k),$$

where $*$ means the product of a vector by a scalar.

Example (2-rank, 6-step)

Let $\mathfrak{g}$ be a 6-step maximal nilpotent Lie algebra of rank 2 and x_1, x_2 be a basis of $\mathfrak{M}_1$. Then $\mathfrak{g}$ is as follows:



and there are following relations:

$22 = 0$ $\cdots\cdots$ (2, 1)

$21 = -12$ $\cdots\cdots$ (2, 2)

$11 = 0$ $\cdots\cdots$ (2, 3)

$2112 = 1212$ $\cdots\cdots$ (4, 1)

$221212 = -122212 + 2 * 212212$ $\cdots\cdots$ (6, 1)

$221112 = -3 * 121212 + 112212 + 3 * 211212$ $\cdots\cdots$ (6, 2)

$211112 = -111212 + 2 * 121112$ $\cdots\cdots$ (6, 3)

The independent vectors of $\mathfrak{M}_2$, $\mathfrak{M}_3$, $\mathfrak{M}_4$, $\mathfrak{M}_5$, $\mathfrak{M}_6$ can be chosen by $\mathfrak{M}'_2$, $\mathfrak{M}'_3$, $\mathfrak{M}'_4$, $\mathfrak{M}'_5$, $\mathfrak{M}'_6$ which are listed below and underlined vectors mean that they depend upon others.

$\mathfrak{M}'_2 = (\underline{22}, 12, \underline{21}, \underline{11})$

There are following relations among these.

$22 = -22 \quad \therefore\ 22 = 0$ $\cdots\cdots$ (2, 1)

$12 = -21 \quad \therefore\ 21 = -12$ $\cdots\cdots$ (2, 2)

$11 = -11 \quad \therefore\ 11 = 0$ $\cdots\cdots$ (2, 3)

$\mathfrak{M}_3' = (212 , 112)$

There are following relations among these.

212=− 〔12 , 2〕

=212 ∵ (2 , 1)

112=− 〔12 , 1〕

=112 ∵ (2 , 2) , (2 , 3)

$\mathfrak{M}_4'$ = (2212 , 1212 , <u>2112</u> , 1112)

There are following relations among these.

2212=− 〔212 , 2〕

=2212 ∵ (2 , 1)

1212=−〔212 , 1〕

=2112−1212+2112 ∵ (2 , 2) , (2 , 3)

∴ 2112=1212 ·· (4 , 1)

1112=−〔112 , 1〕

=1112 ∵ (2 , 3)

$\mathfrak{M}_5'$ = (22212 , 12212 , 21212 , 111212 , 2112 , 11112)

There are following relations among these.

22212=−〔2212 , 2〕

=22212 ∵ (2 , 1)

12212=−〔2212 , 1〕

=21212−21212−21212+12212+21212+21212−21212

∵ (2 , 2),(2 , 5),(4 , 1)

=12212

21212=−〔1212 , 2〕

=21212+12212−12212−21212+21212

∵ (2 , 1),(4 , 1)

=21212

11212=−〔1212 , 1〕

=11212−11212+11212 ∵ (2 , 2),(2 , 3),(4 , 1)

=11212

21112=−〔1112 , 2〕

=11212+11212+11212−11212−11212−11212+21112

∵ (2 , 1),(4 , 1)

=21112

11112=−〔1112 , 1〕

=11112 ∵ (2 , 2),(2 , 3)

$\mathfrak{M}_6'$=(222212 , 122212 , 212212 , 112212 , <u>221212</u> , 121212 ,

211212 , 111212 , <u>221112</u> , 121112 , <u>211112</u> , 111112)

There are following relations among these.

22212=−〔22212 , 2〕

=222212 ∵ (2 , 1)

122212=−〔22212 , 1〕

=221212−221212−221212−221212+212212+212212+221212+212212

+221212+221212−122212−221212−221212−221212+212212

∵ (2 , 2),(2 , 3),(4 , 1)

=−122212+4∗212212−2∗221212

∴ 221212=−122212+2∗212212 ·· (6 , 1)

212212=−〔12212 , 2〕

=221212+122212−212212−212212−221212+122212+221212+221212−212212

∵ (2 , 1),(4 , 1)

=−3∗212212+2∗122212+2∗221212

=212212 ∵ (6 , 1)

112212=−〔12212 , 1〕

=121212−121212−121212+112212+121212+121212−121212

∵ (2 , 2),(2 , 3),(4 , 1)

=112212

121212=−〔21212 , 1〕

=211212−121212−211212+211212+112212+121212+211212−121212

−221112−121212+211212 ∵ (2 , 2),(2 , 3),(4 , 1)

=−2∗121212+112212+3∗211212−221112

∴ 221112=−3∗121212+112212+3∗211212 ··· (6 , 2)

211212=−〔11212 , 2〕

=121212+121212+112212−211212−112212−121212−112212−121212

+121212+221112+121212+121212−211212

∵ (2 , 1),(4 , 1)

=−2∗211212−112212+3∗121212+221112

=211212 ∵ (6 , 2)

111212=−[11212 , 1]

=111212−111212+111212 ∵ (2 , 2),(2 , 3),(4 , 1)

=111212

121112=−[21112 , 1]

=211112−111212+111212+111212+111212−121112−121112−121112

+211112 ∵ (2 , 2),(2 , 3),(4 , 1)

=2∗111212−3∗121112+2∗211112

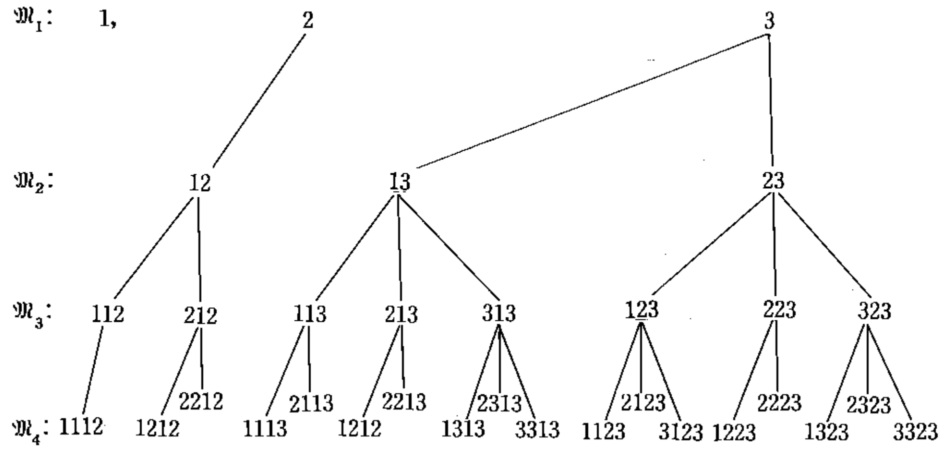
∴ 211112=−111212+2∗121112 ……………………………………(6 , 3)

1111112=−[1112 , 1]

=111112

Example (3-rank, 4-step)

Let $\mathfrak{g}$ be a 4-step maximal nilpotent Lie algebra of rank 3 and x_1, x_2, x_3, a basis of $\mathfrak{M}_1$. Then $\mathfrak{g}$ is as follows:



and there are following relations:

33= 0 ……………………………………(2 , 1)

32=−23 ……………………………………(2 , 2)

31=−13 ……………………………………(2 , 3)

22= 0 ……………………………………(2 , 4)

21=−12 ……………………………………(2 , 5)

11= 0 ……………………………………(2 , 6)

312=−123+213 ……………………………………(3 , 1)

3223=2323 ……………………………………(4 , 1)

3213=1323−3123+2313 ……………………………………(4 , 2)

3212=1223−2∗2123+2213 ……………………………………(4 , 3)

3113=1313 ……………………………………(4 , 4)

3112=−1123+2∗1213−2113 ……………………………………(4 , 5)

2112=−1212 ……………………………………(4 , 6)

The independent vectors of $\mathfrak{M}_2$, $\mathfrak{M}_3$, $\mathfrak{M}_4$ can be chosen from $\mathfrak{M}_2'$, $\mathfrak{M}_3'$, $\mathfrak{M}_4'$ which are listed below and underlined vectors mean that they depend upon others.

$\mathfrak{M}_2'=(\underline{33}, 23, 13, \underline{32}, \underline{22}, 12, \underline{31}, \underline{21}, \underline{11})$

There are following relations among these.

33=−33 ∴ 33= 0 ……………………………………(2 , 1)

23=−32 ∴ 32=−23 ……………………………………(2 , 2)

13=−31 ∴ 31=−13 ……………………………………(2 , 3)

22=−22 ∴ 22= 0 ……………………………………(2 , 4)

12=−21 ∴ 21=−12 ……………………………………(2 , 5)

11=−11 ∴ 11= 0 ……………………………………(2 , 6)

$\mathfrak{M}_3'=(323, 223, 123, 313, 213, 113, \underline{312}, 212, 112)$

There are following relations among these.

323=−[23 , 3]

=323 ∵ (2 , 1)

223=−[23 , 2]

=223 ∵ (2 , 2),(2 , 4)

123=−[23 , 1]

=213−312 ∵ (2 , 3),(2 , 5)

∴ 312=−123+213 ……………………………………(3 , 1)

313=−[13 , 3]

=313 ∵ (2 , 1)

213=−[13 , 2]

=123+312 ∵ (2 , 2)

=213 ∵ (3 , 1)

113=−[13 , 1]

=113 ∵ (2 , 6)

212=−[12 , 2]

=212 ∵ (2 , 4)

112=−[12 , 1]

=112 ∵ (2 , 6)

$\mathfrak{M}_4' = (3323\ ,\ 2323\ ,\ 1323\ ,\ \underline{3223}\ ,\ 2223\ ,\ 1223\ ,\ 3123\ ,\ 2123\ ,$
$1123\ ,\ 3313\ ,\ 2313\ ,\ 1313\ ,\ \underline{3213}\ ,\ 2213\ ,\ 1213\ ,\ \underline{3313}\ ,$
$2113\ ,\ 1113\ ,\ \underline{3212}\ ,\ 2212\ ,\ 1212\ ,\ \underline{3112}\ ,\ \underline{2112}\ ,\ 1112\ )$

There are following relations among these.

3323 = −〔323 , 3〕
 = 3323 ∵ (2 , 1)
2323 = −〔323 , 2〕
 = 3223 − 2323 + 3223 ∵ (2 , 2)
 = −2323 + 2 * 3223
∴ 3223 = 2323 ……………………………… (4 , 1)
1323 = −〔323 , 1〕
 = 3213 − 2313 + 3123 − 3213 + 3213 ∵ (2 , 3), (3 , 1)
 = 3123 − 2313 + 3213
∴ 3213 = 1323 − 3123 + 2313 ……………………………… (4 , 2)
2223 = −〔223 , 2〕
 = 2223 ∵ (2 , 2)
1223 = −〔223 , 1〕
 = 2213 + 2123 − 2213 + 2123 − 2213 + 3212
 ∵ (2 , 3), (2 , 5), (3 , 1)
 = 2 * 2123 − 2213 + 3212
∴ 3212 = 1223 − 2 * 2123 + 2213 ……………………………… (4 , 3)
3123 = −〔123 , 3〕
 = 2313 + 1323 − 3213 ∵ (2 , 1)
 = 3123 ∵ (4 , 2)
2123 = −〔123 , 2〕
 = 1223 − 2123 + 2213 − 3212 ∵ (2 , 2), (3 , 1)
 = 2123 ∵ (4 , 3)
1123 = −〔123 , 1〕
 = 1213 + 1123 − 1213 ∵ (2 , 3), (2 , 5), (2 , 6), (3 , 1)
 = 1123
3313 = −〔313 , 3〕
 = 3313 ∵ (2 , 1)
2313 = −〔313 , 2〕
 = 3123 − 1323 − 3123 + 3213 + 3123 ∵ (2 , 2) , (3 , 1)

 = 2313 ∵ (4 , 2)
1313 = −〔313 , 1〕
 = 3113 − 1313 + 3113 ∵ (2 , 3), (2 , 6)
∴ 3113 = 1313 ……………………………… (4 , 4)
2213 = −〔213 , 2〕
 = 2123 − 2123 + 2213 ∵ (2 , 2), (2 , 4), (3 , 1)
 = 2213
1213 = −〔213 , 1〕
 = 2113 + 1123 − 1213 + 3112 ∵ (2 , 3), (2 , 5), (3 , 1)
 = 1123 − 1213 + 2113 + 3112
∴ 3112 = −1123 + 2 * 1213 − 2113 ……………………………… (4 , 5)
2113 = −〔113 , 2〕
 = 1123 − 1123 + 1213 − 1123 + 1213 − 3112
 ∵ (2 , 2), (3 , 1)
 = 2113
1113 = −〔113 , 1〕
 = 1113 ∵ (2 , 3)
2212 = −〔212 , 2〕
 = 2212 ∵ (2 , 4)
1212 = −〔212 , 1〕
 = 2112 − 1212 + 2112 ∵ (2 , 5), (2 , 6)
∴ 2112 = −1212 ……………………………… (4 , 6)
1112 = −〔112 , 1〕 ∵ (2 , 5)
 = 1112

§ 5. Classification

In this section, we shall apply the results of Sec. 2 to the 4-step nilpotent Lie algebras of rank 2 and classify them.

Difinition 4 . *Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras and f a linear mapping of $\mathfrak{g}$ onto $\mathfrak{h}$. The mapping f is called an isomorphism if the following conditions are satisfied* :

1) $f([X, Y]) = [f(X), f(Y)]$ $X, Y \in \mathfrak{g}$,
2) If $X \neq Y$, then $f(X) \neq f(Y)$,

and then these algebras are said to be isomorphic.

If $\mathfrak{g}$ and $\mathfrak{h}$ are especially nilpotent Lie algebras, then we can look for them to be contained within a maximal one with the same rank and step (as subalgebras).

Lemma 3. *Let* $\mathfrak{g}$, $\mathfrak{h}$ *be a* k*-step nilpotent Lie algebras of rank* n, *and* x_{11}, x_{12}, $\cdots\cdots$, x_{1n} *the common generators in maximal one. Then* $\mathfrak{g}$ *and* $\mathfrak{h}$ *are isomorphic if and only if there are* $n\times n$ *and* $n\times(N-n)$ *matrices* A $(|A|\neq 0)$ *and* B *such that the vectors*

$$\begin{pmatrix} x_{11}^* \\ x_{12}^* \\ \vdots \\ x_{1n}^* \end{pmatrix} = (AB)\begin{pmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{kn_k} \end{pmatrix}, \text{where } x_{11}, x_{12}, \cdots\cdots x_{kn_k} \text{ is a basis of } \mathfrak{h}$$

and their descended vectors satisfy the same linear relations as $\mathfrak{g}$*'s one.*

Proof

If $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic, then there is an isomorphism f which satisfies 1) and 2) and since x_{1i} ($i=1,2,\cdots,n$) are generators of $\mathfrak{g}$, $x^*_{1i}=f(x_{1i})$ ($i=1,2,\cdots,n$) are also generators of $\mathfrak{h}$. Therefore there are $n\times n$ and $n\times(N-n)$ matrices A ($|A|\neq 0$) and B respecively such that

$$(*) \qquad \begin{pmatrix} x_{11}^* \\ x_{12}^* \\ \vdots \\ x_{1n}^* \end{pmatrix} = (AB)\begin{pmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{kn_k} \end{pmatrix}$$

and their descended vectors satisfiy $\mathfrak{g}$'s linear relations.

Conversely if these relations are satisfied, then linear mapping which is induced from relation $(*)$,

$$\mathfrak{g} : \begin{pmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{kn_k} \end{pmatrix} \to \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{kn_k} \end{pmatrix}$$

is a isomorphism $\mathfrak{h}$ onto $\mathfrak{g}$.

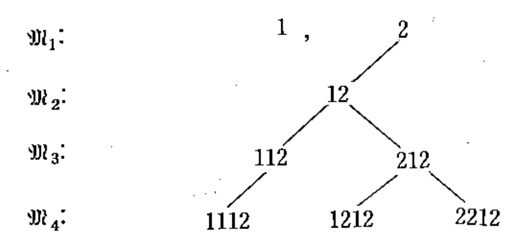
Remark Q. E. D.

General linear group GL $(n,\mathbf{R})$ has an Iwasawa decomposition. That is, GL $(n,\mathbf{R}) = O(n)D(n)N(n)$ where $O(n)$, $D(n)$ and $N(n)$ are all $n\times n$ matrices of orthogonal, real diagonal and upper triangular with 1's on the diagonal respectively.

Therefore we can classify the nilpotent Lie algebras by mean of the linear mappings $f = (N(n)\ 0)$, $(D(n)\ 0)$, $(O(n)\ 0)$ and $(I\ M)$ only, where M is all $n\times(N-n)$ matrices.

According to the above remark, we shall classify the 4-step nilpotent Lie algebras of rank 2 as follows:

The maximal nilpotent Lie algeara $\bar{\mathfrak{g}}$ is



and there are following relations:

$22 = 0$ $\cdots\cdots$ (2, 1)

$21 = -12$ $\cdots\cdots$ (2, 2)

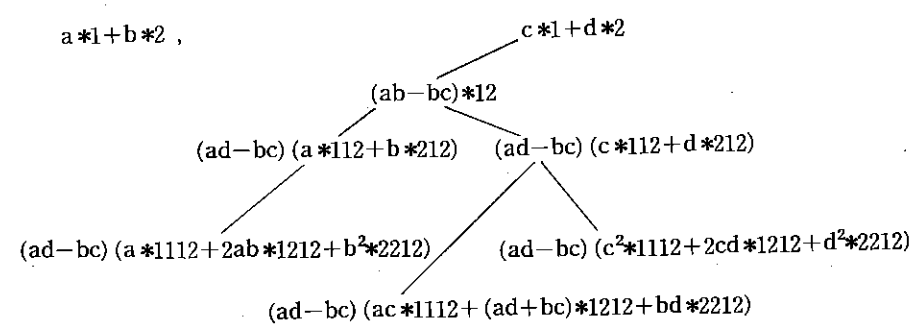
$11 = 0$ $\cdots\cdots$ (2, 3)

$2112 = -1212$ $\cdots\cdots$ (4, 1)

We transform the generators x_1, x_2 to another generators x_1^*, x_2^* by the linear relation

$$(1) \qquad \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$$

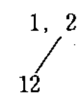
then the maximal Lie algebra with generators x_1^*, x_2^* is



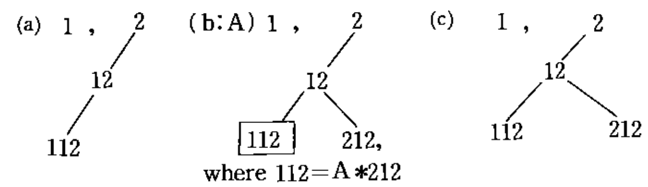
We can find all non-isomorphic Lie algebras by these relations.

1-step: 1, 2

2-step:



3-step:



If 3-step nilpotent Lie algebras $\mathfrak{g}$ are transformed to the type (a) by the relation (1), then we get

$$c*112+d*212=0.$$

When $c\neq 0$, then

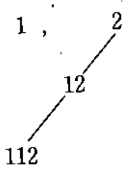
$$112=-\frac{d}{c}*212.$$

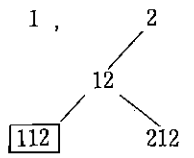
Therefore

$$A(\text{in the type (b)})=-\frac{d}{c} \text{ and } -\infty<d<\infty,$$

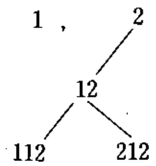
i. e., this means that (a) $\cong$ (b)

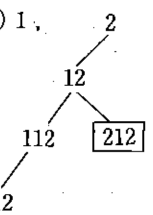
Hence all non-isomorphic Lie algebras are following (a) and (c):

(a) 

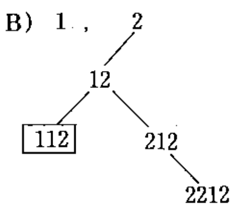
(b:A) 

where $\boxed{112}=A*212$

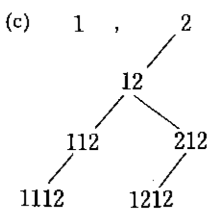
(c) 

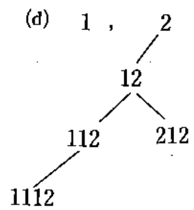
4-step (a : A) 

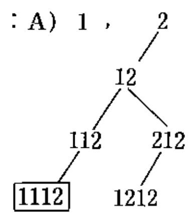
where $\boxed{212}=A*1112$

(b ; A, B) 

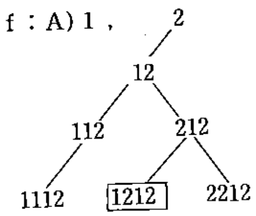
where $\boxed{112}=A*212+B*2212$

(c) 

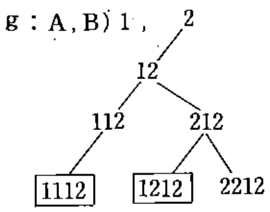
(d) 

(e : A) 

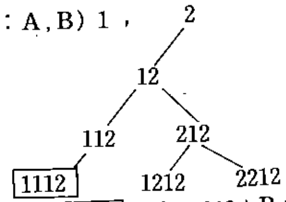
where $\boxed{1112}=A*1212$

(f : A) 

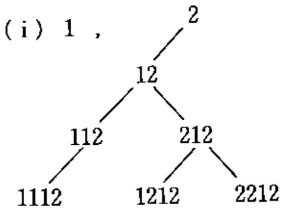
where $\boxed{1212}=A*2212$

(g : A, B) 

where $\boxed{1212}=A*2212$ and $\boxed{1112}=B*2212$

(h : A, B) 

where $\boxed{1112}=A*1212+B*2212$

(i) 

We first consider the case that 4-step nilpotent Lie algebras $\mathfrak{g}$ are transformed to the following types by the relation (1),

i) Type (a : A)

We get

$$c*112+d*212=A(a^2*1112+2ab*1212+b^2*2212). \qquad \text{(a , 1)}$$

When $c=0$, then $d\neq 0$ and

$$212=\frac{A}{d}(a^2*1112+2ab*1212+b^2*2212)$$

$$=\frac{Aa^2}{d}*1112 \qquad \because\ 1212=2212=0.$$

Therefore

$$A(\text{in the type (a)})=\frac{Aa^2}{d} \text{ and } a\neq 0$$

i. e., this means that $(a : A=1)\cong(a : A\neq 0)$.

When $c\neq 0$, then

$$112=-\frac{d}{c}*212+\frac{A}{c}*(a^2*1112+2ab*1212+b^2*2212)$$

$$=-\frac{d}{c}*212+\frac{A(ad-bc)^2}{c^3}*2212 \qquad \because\ 1112=-\frac{d}{c}*1212,$$

$$1212=2112=-\frac{d}{c}*2212.$$

Therefore

$$A(\text{in the type (b)})=-\frac{d}{c}, B(\text{in the type (b)})=\frac{A(ad-bc)^2}{c^3} \text{ and } -\infty<d<\infty, \text{ i. e.,}$$

this means that $(a : A=0)\cong(b : B=0)$ and $(a : A=1)\cong(b : B\neq 0)$.

ii) Type(c)

We get

$c^2*1112+2cd*1212+d^2*2212=0$. ·························(c, 1)

When $c=0$, then

$d\neq 0$, and $2212=0$, i. e., it's of the same type.

When $c\neq 0$, then

$1112=-\frac{2d}{c}*1212-\frac{d^2}{c^2}*2212.$

Therefore

A(in the type (h)) $=-\frac{2d}{c}$, B (in the type (h)) $=-\frac{d^2}{c^2}$ and $-\infty<d<\infty$.

$\therefore A^2=-4B$,

i. e., this means that (c) $\cong$ (h : $A^2+4B=0$).

iii) Type (d)

We get

$c^2*1112+2cd*1212+d^2 2212=0$, ·························(d, 1)

$ac*1112+(ad+bc)*1212+bd*2212=0$. ·························(d, 2)

When $c=0$, then

$d\neq 0$ and $1212=2212=0$, i. e., it's of the same type.

When $c\neq 0$, from (d, 1), (d, 2) it follows

$2acd*1212+ad^2*2212-(ad+bc)c*1212-bcd*2212=0.$

$\therefore 1212=-\frac{d}{c}*2212.$

From (d, 1), it follows

$1112=\frac{d^2}{c^2}*2212.$

Therefore

A(in the type (g)) $=-\frac{d}{c}$, B(in the type (g)) $=\frac{d^2}{c^2}$ and $-\infty<d<\infty$.

$\therefore A^2=B$,

i. e., this means that (d) $\cong$ (g : $B=A^2$).

iv) Type (e : A)

We get

$c^2*1112+2cd*1212+d^2*2212=0$, ·························(e, 1)

$a^2*1112+2ab*1212+b^2*2212=A(ac*1112+(ad+bc)*1212+bd*2212)$. ······(e, 2)

When $c=0$ (resp. $a=0$) and $A=0$, then $a\neq 0$, $d\neq 0$ (resp. $b\neq 0$, $c\neq 0$)

From (e, 1), (e, 2) it follows

$2212=0$ and $1112=-\frac{2b}{a}*1212$ (resp. $-\frac{2d}{c}*1212$).

Therefore A (in the type (e)) $=-\frac{2b}{a}$(resp. $-\frac{2d}{c}$) and $-\infty<d<\infty$ (resp. $-\infty<d<\infty$),

i. e., this means that (e : $A=0$) $\cong$ (e : $A\neq 0$)

When $c\neq 0$, $a\neq 0$ and $A=0$,

From (e : 1), (e, 2) it follows

$-2a^2cd*1212-a^2d^2*2212+2abc^2*1212+b^2c^2*2212=0.$

$\therefore 2ac*1212+(ad+bc)*2212=0.$

$\therefore 1212=-\frac{ad+bc}{2ac}*2212.$

From (e, 1) it follows

$1112=\frac{bd}{ac}*2212.$

Therefore

A(in the type (g)) $=-\frac{ad+bc}{2ac}$, B (in the type (g)) $=\frac{bd}{ac}$

$\therefore A^2-B=\frac{(ad-bc)^2}{4a^2c^2}>0$ $\therefore ad-bd\neq 0$,

i. e., this means that (e : $A=0$) $\cong$ (g : $B<A^2$).

v) Type (f : A)

We get

$ac*1112+(ad+bc)*1212+bd*2212=A(c^2*1112+2cd*1212+d^2*2212)$ ········(f, 1)

When $c=0$ (resp. $a=0$) and $A=0$, then

$a\neq 0$, $d\neq 0$ (resp. $b\neq 0$, $c\neq 0$) and $1212=-\frac{b}{a}*2212$ (resp. $-\frac{d}{c}*2212$).

Therefore

$A=-\frac{b}{a}$ (resp. $-\frac{d}{c}$) and $-\infty<b<\infty$ (resp. $-\infty<d<\infty$),

i. e., this means that (f : $A=0$) $\cong$ (f : $A\neq 0$).

When $c\neq 0$, $a\neq 0$ and $A=0$, then

$1112=-\frac{ad+bc}{ac}*1212-\frac{bd}{ac}*2212$

Therefore

A (in the type (h)) $=-\frac{ad+bc}{ac}$, B (in the type (h)) $=-\frac{bd}{ac}$.

$\therefore A^2+4B=\frac{(ad-bc)^2}{a^2c^2}>0$ $\therefore ad-bc\neq 0$,

i. e., this means that (f : $A=0$) $\cong$ (h : $A^2+4B>0$).

vi) Type (g : $B>A^2$)

We get

$ac*1112+(ad+bc)*1212+bd*2212=A(c^2*1112+2cd*1212+d^2*2212)$, ········(g, 1)

$a^2*1112+2ab*1212+b^2*2212=B(c^2*1112+2cd*1212+d^2*2212)$. ················(g, 2)

When $A=0$, $B=1$ and $c=0$, then $a\neq 0$ and

From (g, 1) (g, 2) if follows

$1212=-\frac{b}{a}*2212$ and $1112=\frac{b^2+d^2}{a^2}*2212.$

Therefore A (in the type (g)) $=-\frac{b}{a}$, B(in the type (g)) $=\frac{b^2+d^2}{a^2}$

$$\therefore\ B-A^2=\frac{d^2}{a^2}>0 \qquad \therefore d\neq 0\,.$$

i. e., this means that $(g: A=0,\ B=1)\cong(g: B>A^2)$.

vii) Type $(h: A^2+4B<0)$

We get

$$a^2*1112+2ab*1212+b^2*2212 = A(ac*1112+(ad+bc)*1212+bd*2212)$$
$$+B(c^2*1112+2cd*1212+d^2*2212. \cdots\cdots\cdots\cdots (h,\ 1)$$

When $A=0,\ B=-1$, then

$$a^2*1112+2ab*1212+b^2*2212 = -c*1112-2cd*1212-d^2*2212.$$

$$\therefore\ 1112=-\frac{2(ab+cd)}{a^2+c^2}*1212-\frac{b^2+d^2}{a^2+c^2}*2212.$$

Therefore A (in the type (h)) $=-\dfrac{2(ab+cd)}{a^2+c^2}$, B (in the type (h)) $=-\dfrac{b^2+d^2}{a^2+c^2}$.

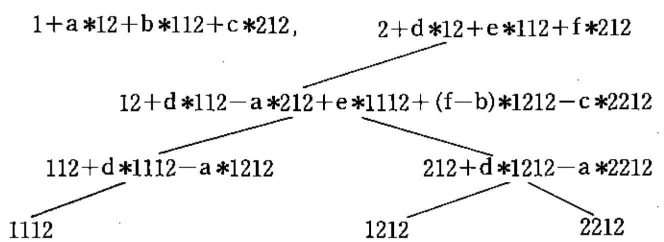
$$\therefore\ A^2+4B=-\frac{4(ad-bc)^2}{(a^2+c^2)^2}<0 \qquad \therefore ad-bc\neq 0\,,$$

i. e., this means that $(h: A=0,\ B=-1)\cong(h: A^2+4B<0)$.

Next, we consider the case that 4-step nilpotent Lie algebras g are transformed to the following types by the linear relation

$$(2)\qquad \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}=\begin{pmatrix} 1 & 0 & a & b & c \\ 0 & 1 & d & e & f \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_1x_2 \\ x_1x_1x_2 \\ x_2x_1x_2 \end{pmatrix},$$

then the maximal Lie algebra is



i) Type (a : A = 0)

We get

$$212+d*1212-a*2212=0$$

$$\therefore\ 212=-d*1212+a*2212=0 \quad \because\ 1212=2212=0,$$

i. e., it's of the same type.

ii) Type (a : A = 1)

We get

$$212+d*1212-a*2212=1112$$

$$\therefore\ 212=1112-d*1212+a*2212=1112 \quad \because\ 1212=2212=0,$$

i . e., it's of the same type.

iii) Type (c)

We get

$2212=0$, i . e., it's of the same type.

iv) Type (d)

We get

$1212=2212=0$, i. e., it's of the same type.

v) Type (e : A = 0)

We get

$1112=2212=0$, i. e. , it's of the same type.

vi) Type (f : A = 0)

We get

$1212=0$, i. e., it's of the same type.

vii) Type (g : A = 0, B = 1)

We get

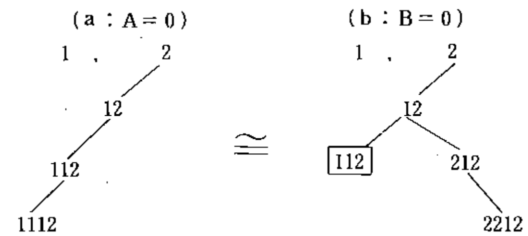
$1112=2212$, i. e., it's of the same type.

viii) Type (h : A = 0, B = −1)

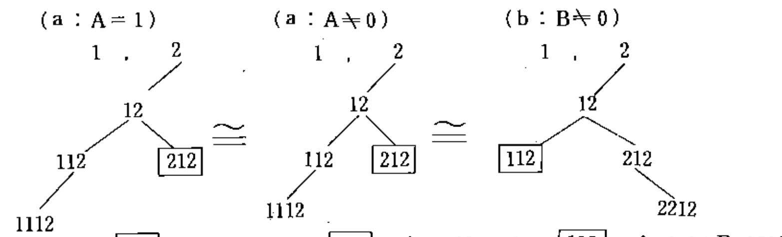
We get

$1112=-2212$, i. e., it's of the same type.

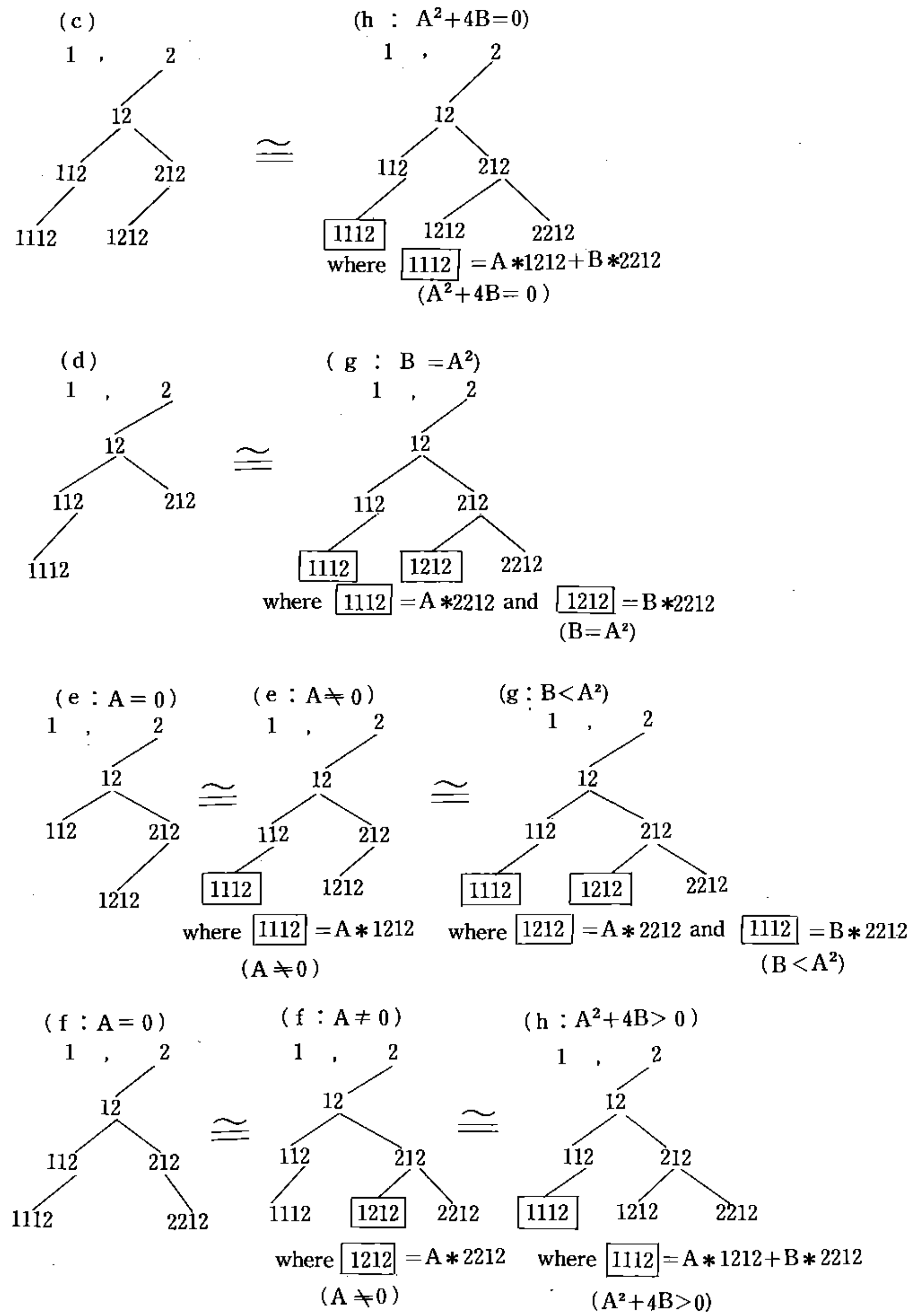
Hence all non-isomorphic Lie algebras are as follows :

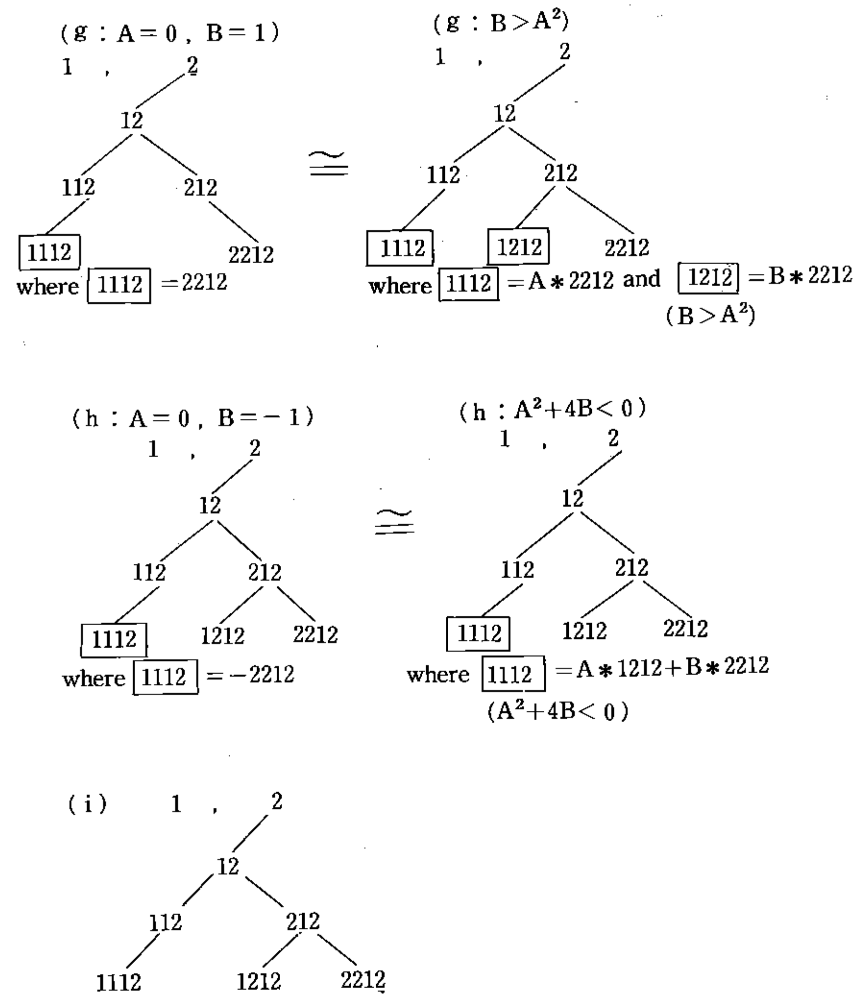


where $\boxed{112}=A*212$



where $\boxed{212}=1112$ where $\boxed{212}=A*1112$ $(A\neq 0)$ where $\boxed{112}=A*212+B*2212$ $(B\neq 0)$





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