A NOTE OF AFFINE STRUCTURES OF MAXIMAL SOLVABLE SUBALGEBRAS OF NONCOMPACT SEMISIMPLE LIE ALBEBRAS

By

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1. Introduction. The aim of this note is to prove the following

Theorem. Let $\mathfrak{b}$ and $\mathfrak{b}_0$ be the following solvable Lie subalgebras :

(1) a Borel subalgebra $\mathfrak{b}$ of a complex semisimple Lie algebra $\mathfrak{g}$.

(2) a maximal solvable subalgebra $\mathfrak{b}_0 = \mathfrak{h}^- + \mathfrak{a}_{\mathfrak{p}} + \mathfrak{n}^+$ of a noncompact real semisimple Lie algebra $\mathfrak{g}_0$.

Then $\mathfrak{b}$, $\mathfrak{b}_0$ and their suitable sualgebras admit affine structures.

In the above and in what follows, we say for short that a solvable Lie algebra $\mathfrak{s}$ over R admits an affine structure if a connected Lie group S with Lie algebra $\mathfrak{s}$ admits a complete locally flat affine structure which is invariant under left translation, or equivalently, the universal covering group $\widetilde{S}$ operates simply transitively by affine transformations of R^n, where $n = \dim \mathfrak{s}$.

The idea of proof is as follows, Both algebras satisfy the same properties as stated in §1 of [2];

(1) $\mathfrak{s} = \mathfrak{n} + \mathfrak{h}$ (semidirect sum), where $\mathfrak{s} = \mathfrak{b}$ or $\mathfrak{b}_0$.

(2) $\mathfrak{n}$ is a nilpotent ideal which is graded by positive integers.

(3) $\mathfrak{h}$ is abelian and preserves the gradation of $\mathfrak{n}$.

So $\mathfrak{s}$ admits an affine structure by the Theorem of [2].

2. Proof of Theorem.

In what follows, refer to [1] for the details on $\mathfrak{g}$ and $\mathfrak{g}_0$.

Case 1. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ and $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ the set of all nonzero roots with respect to $\mathfrak{h}$. As usual, we introduce a lexicographic order and let Δ^+ (resp. $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$) be the set of all positive roots (the fundamental system of roots) with respect to this order. Put $\mathfrak{n}^+ = \sum_{\alpha \in \Delta^+} \mathfrak{g}^\alpha$, where $\mathfrak{g}^\alpha$ denotes a root space corresponding to $\alpha \in \Delta^+$. Then $\mathfrak{b} = \mathfrak{n}^+ + \mathfrak{h}$ is called a Borel (or minimal parabolic) subalgebra of $\mathfrak{g}$. Each $\alpha \in \Delta^+$ is written uniquely as $\alpha = \sum_{k=1}^{\ell} m_k \alpha_k$ ($m_k \geq 0$, integers). Now we define a gradation of $\mathfrak{n}^+$ by setting $\mathfrak{n}_i = \sum_{|\alpha|=i} \mathfrak{g}^\alpha$, where we put $|\alpha| = \sum_{k=1}^{\ell} m_k$ for $\alpha = \sum_{k=1}^{\ell} m_k \alpha_k \in \Delta^+$. Then we have :

Lemma. Let I be the set $\{ i = \sum_{k=1}^{\ell} m_k ; \sum_{k=1}^{\ell} m_k \alpha_k \in \Delta^+ \}$. Then $\{ \mathfrak{n}_i ; i \in I \}$ gives a gradation of $\mathfrak{n}^+$ by positive integers and $\mathfrak{h}$ preserves the gradation, i.e., $\mathfrak{n}^+ = \sum_{i \in I} \mathfrak{n}_i$ (direct sum), $[\mathfrak{n}_i, \mathfrak{n}_j] = \mathfrak{n}_{i+j}$ and $\mathrm{ad}(\mathfrak{h}) \cdot \mathfrak{n}_i \subseteq \mathfrak{n}_i$.

Proof. Since $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \subseteq \mathfrak{g}^{\alpha+\beta}$ and $\mathrm{ad}(\mathfrak{h}) \cdot \mathfrak{g}^\alpha \subseteq \mathfrak{g}^\alpha$, Lemma is obvious.

Therefore $\mathfrak{b} = \mathfrak{n} + \mathfrak{h}$ satisfies the properties stated in §1. Consequently $\mathfrak{b}$ admits a complex affine structure. As a real Lie algebra $\mathfrak{b}^{\mathbb{R}}$, it also admits a real affine structure (cf. Proof of Corollary 1, [2]).

Case 2. Let $\mathfrak{g}_o$ be a noncompact real semisimple Lie algebra. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be a Cartan decomposition, $\mathfrak{a}_\mathfrak{p}$ a maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{m}=\mathfrak{c}_\mathfrak{k}(\mathfrak{a}_\mathfrak{p})$ the centralizer of $\mathfrak{a}_\mathfrak{p}$ in $\mathfrak{k}$. let $\mathfrak{a}_\mathfrak{p}^*$ be the dual space of $\mathfrak{a}_\mathfrak{p}$. For each $\lambda\in\mathfrak{a}_\mathfrak{p}^*$, we set $\mathfrak{g}_o^\lambda=\{X\in\mathfrak{g}_o\ ;\ [A, X]=\lambda(A)X \text{ for all } A\in\mathfrak{a}_\mathfrak{p}\}$. $\lambda\in\mathfrak{a}_\mathfrak{p}^*$ such that $\lambda\neq 0$ and $\mathfrak{g}_o^\lambda\neq 0$ is called a restricted root. Let $\Sigma=\Sigma(\mathfrak{g}_o,\mathfrak{a}_\mathfrak{p})$ be the set of all restricted roots. Then Σ forms a root system and each root system has a basis, i.e., there exist $\lambda_1,\dots,\lambda_r\in\Sigma$, a basis of $\mathfrak{a}_\mathfrak{p}^*$, such that each $\lambda\in\Sigma$ can be written uniquely as

$$\lambda=\sum_{k=1}^{r} m_k\lambda_k,$$

where the m_k are integers of the same sign. Now we put

$$\Sigma^+=\{\lambda=\sum_{k=1}^{r} m_k\lambda_k\in\Sigma\ ;\ m_k\geq 0\},\quad \mathfrak{n}^+=\sum_{\lambda\in\Sigma^+}\mathfrak{g}_o^\lambda,$$
$$I=\{|\lambda|=\sum_{k=1}^{r} m_k\ ;\ \lambda=\sum_{k=1}^{r} m_k\lambda_k\in\Sigma^+\},\quad \mathfrak{n}_i=\sum_{|\lambda|=i}\mathfrak{g}_o^\lambda.$$

Then as in the Case 1, $\mathfrak{n}^+$ is graded by I. Let $\mathfrak{h}^-$ be a maximal abelian sualgebra of $\mathfrak{m}$ and put $\mathfrak{b}_o=\mathfrak{h}^-+\mathfrak{a}_\mathfrak{p}+\mathfrak{n}^+$. Then $\mathfrak{b}_o=\mathfrak{n}^+ +_s \mathfrak{h}$ (semidirect sum) is a maximal solvable subalgebra of $\mathfrak{g}_o$, where we put $\mathfrak{h}=\mathfrak{h}^-+\mathfrak{a}_\mathfrak{p}$. Then it is easy to see that the same Lemma as in the Case 1 holds also in this case. Consequently $\mathfrak{b}_o$ admits an affine structure by the Theorem of [2].

Summing up, the Theorem is proved.

Remark. From the above argument, it is clear that any subalgebra $\mathfrak{s}$ of the form $\mathfrak{s}=\mathfrak{n}' +_s \mathfrak{h}'$ admits an affine structure, where $\mathfrak{h}'$ is any subalgebra of $\mathfrak{h}$ and I' is any subset of I such that $\mathfrak{n}'=\sum_{i\in I'}\mathfrak{n}_i$ becomes a subalgebra.

References

[1] S. Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces, Academic Press, New York, 1978.

[2] S. Yamaguchi, Affine structure of some solvable Lie group, Mem. Fac. Sci. Kyushu Univ., to appear.